Fachbereich Informatik und
Informationswissenschaft

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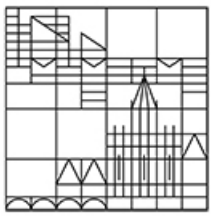

Lecture Notes

Network Dynamics

taught in Winter terms 2010, 2011, 2012, 2013, 2014

by

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Contents

An Internet example 1

Internet routing involves the *next-hop* principle in the following way:

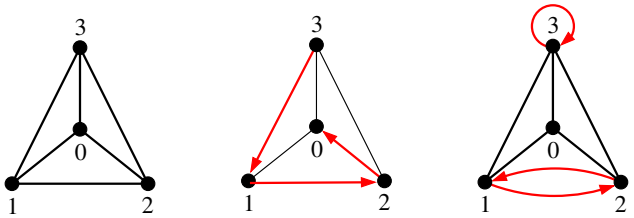
- A *path* is a sequence of entities which pairwise share a direct data link.
- Sending data from an entity to an entity (the next hop) over a direct data link is called *forwarding*.
- A *transmission* is the process of successively forwarding data from a source to a destination over a path with source as its first and the destination as its last element.
- The selection of a path for a transmission is called *routing*.

1.1 Policy routing

We consider the formal next-hop routing problem for a destination node 0.

Suppose we are given an undirected (or bidirected) graph $G = (V, E)$, i.e., it holds that $(u, v) \in E \Leftrightarrow (v, u) \in E$. Each nodes chooses a neighbor in G including itself. A *configuration* is a mapping $\pi : V \to V : u \mapsto v \in N^+(u) \cup \{u\}$.

> **Example:** Assume that a tiny portion of the Internet is represented by the following graph $G = (V, E)$ and configurations are represented by red arrows:



$\pi(1) = 2$
$\pi(2) = 0$
$\pi(3) = 1$

good

$\pi(1) = 2$
$\pi(2) = 1$
$\pi(3) = 3$

bad

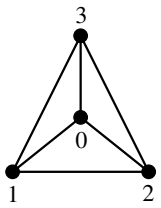
The mutual goal is to find a directed tree rooted at 0. We say that a configuration π is *confluent* if and only if for each $u \in V$, there is a $k \in \mathbb{N}$ such that $\pi^k(u) = 0$. Recall that the function $\pi^k : V \to V$ is inductively defined by $\pi^0(x) =_{\text{def}} x$ and $\pi^k(x) =_{\text{def}} \pi(\pi^{k-1}(x))$

for all $x \in V$ and $k \in \mathbb{N}$. It is clear that a configuration π is confluent if and only if the directed graph $(V, \{(x, \pi(x)) | x \in V \setminus \{0\}\})$ is a tree rooted at 0. Note that a directed graph $G = (V, E)$ is called *tree* (rooted at $v \in V$) iff $d^+(v) = 0$ and there is exactly one (u, v)-path for all $u \in V$. A good choice of a tree is a shortest-path tree; however, it needs global coordination.

An approach which avoids global coordination makes use of the following notions:

- $P^u =_{\text{def}} \{\ p \mid p \text{ is a path in } G \text{ starting at } u \text{ and ending at } 0\ \} \cup \{(u)\}$ for each $u \in V$
- $(P^u, \succeq_u)$ is a total preorder (preference ordering, ranking)
- $p \succ_u (u)$ for all $p \in P^u \setminus \{(u)\}$ (i.e., non-reachability is the last resort or reachability is always superior to non-reachability)

Example: We extend our example by ranked path sets P^u.



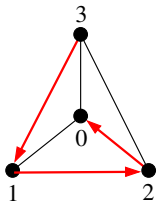
$$
\begin{aligned}
P^0 &: (0) \\
P^1 &: (1,0) \succ_1 (1,2,0) \succ_1 (1,3,0) \succ_1 (1,3,2,0) =_1 (1,2,3,0) \succ_1 (1) \\
P^2 &: (2,3,0) \succ_2 (2,1,0) \succ_2 (2,1,3,0) \succ_2 (2,3,1,0) \succ_2 (2,0) \succ_2 (2) \\
P^3 &: (3,1,0) \succ_3 (3,1,2,0) \succ_3 (3,0) \succ_3 (3,2,0) \succ_3 (3,2,1,0) \succ_3 (3)
\end{aligned}
$$

For instance, node 3 always prefers a route over node 1 over a direct route to zero or a route over node 2.

Now, suppose a given configuration is

$$\pi(0) = 0, \quad \pi(1) = 2, \quad \pi(2) = 0, \quad \pi(3) = 1;$$

thus the nodes realize the following paths:



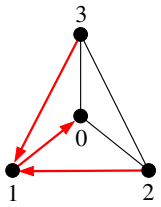
$$
\begin{aligned}
P^0 &: (0) \\
P^1 &: (1,0) \succ_1 (1,2,0) \succ_1 (1,3,0) \succ_1 (1,3,2,0) =_1 (1,2,3,0) \succ_1 (1) \\
P^2 &: (2,3,0) \succ_2 (2,1,0) \succ_2 (2,1,3,0) \succ_2 (2,3,1,0) \succ_2 (2,0) \succ_2 (2) \\
P^3 &: (3,1,0) \succ_3 (3,1,2,0) \succ_3 (3,0) \succ_3 (3,2,0) \succ_3 (3,2,1,0) \succ_3 (3)
\end{aligned}
$$

An easy analysis shows that node 1 can realize path $(1, 0)$ by using its directed connection to 0, thus a better alternative; node 2 can realize an other option without destroying confluence; node 3 cannot realize path $(3, 1, 0)$. Therefore, the configuration is not stable.

If the given configuration is instead

$$\pi(0) = 0, \quad \pi(1) = 0, \quad \pi(2) = 1, \quad \pi(3) = 1,$$

nodes can realize the following paths to 0:



$$\begin{aligned}
P^0 &: (0) \\
P^1 &: (1,0) \succ_1 (1,2,0) \succ_1 (1,3,0) \succ_1 (1,3,2,0) =_1 (1,2,3,0) \succ_1 (1) \\
P^2 &: (2,3,0) \succ_2 (2,1,0) \succ_2 (2,1,3,0) \succ_2 (2,3,1,0) \succ_2 (2,0) \succ_2 (2) \\
P^3 &: (3,1,0) \succ_3 (3,1,2,0) \succ_3 (3,0) \succ_3 (3,2,0) \succ_3 (3,2,1,0) \succ_3 (3)
\end{aligned}$$

Again, an easy analysis shows that nodes 1 and 3 both realize their best alternatives and node 2 cannot realize path $(2,3,0)$, so node 2 realizes also its best path given all other paths. Threfore, the configuration can be viewed as stable.

The example suggests the following definition: A configuration π is said to be *stably confluent* iff π is confluent and no node u can realize a better path to 0 without losing confluence. A stably confluent configuration solves the formal routing problem.

1.2 Best-response dynamics

But how can we find a stably confluent configuration without global coordination? The mechanism is: always take the best path. We give a formal description of the mechanism, which represents a very simplified BGP dynamics.

Let $\pi : V \to V$ be any configuration for a graph $G = (V, E)$. For a node $u \in V$, define the following path set

$$R_{u,0,\pi} =_{\text{def}} \{\ p(v) \quad | \quad v \in N^+(u) \text{ and } p(v) \text{ is a } (u,0)\text{-path with next hop } v \text{ in graph } (V, \{(x, \pi(x)) | x \neq u\} \cup \{(u,v)\}\ \}$$

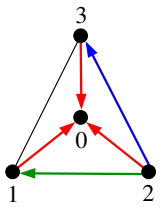
and the following function

$$\beta_u(\pi) =_{\text{def}} \begin{cases} v & \text{if } p(v) = \max_{\succ_u} R_{u,0,\pi} \\ u & \text{if } R_{u,0,\pi} = \emptyset \end{cases}$$

Here we assume, for simplicity of description, that $(P^u, \succeq_u)$ is total order such that $p \succ_u (u)$. Finally, define $\beta(\pi) : u \mapsto \beta_u(\pi)$.

Example (cont'd): Consider the graph and the sets of paths from above. Suppose all nodes update their paths simultaneously. Then, we could obtain

the following evolution of the routing network when starting with the inital configuration in time step $t = 0$:

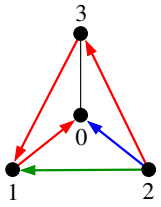


$$
\begin{aligned}
P^0 &: (0)\\
P^1 &: (1,0) \succ_1 (1,2,0) \succ_1 (1,3,0) \succ_1 (1,3,2,0) =_1 (1,2,3,0) \succ_1 (1)\\
P^2 &: (2,3,0) \succ_2 (2,1,0) \succ_2 (2,1,3,0) \succ_2 (2,3,1,0) \succ_2 (2,0) \succ_2 (2)\\
P^3 &: (3,1,0) \succ_3 (3,1,2,0) \succ_3 (3,0) \succ_3 (3,2,0) \succ_3 (3,2,1,0) \succ_3 (3)
\end{aligned}
$$

If we consider node 2 then the path colored red in the path set is best path (only path) with next hop 0, the path colored green is the best path with next hop 1, and the path colored blue is the best path with next hop 2. That is, node 2 choose 2 as the next hop. Similar considerations for the other nodes lead to the following new configuration

$$\beta_0(\pi) = 0, \quad \beta_1(\pi) = 0, \quad \beta_2(\pi) = 3, \quad \beta_3(\pi) = 1.$$

After one update step of each nodes given the network in $t = 0$, the resulting routing network is thus the following ($t = 1$):

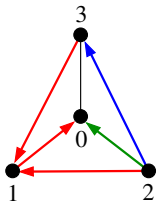


$$
\begin{aligned}
P^0 &: (0)\\
P^1 &: (1,0) \succ_1 (1,2,0) \succ_1 (1,3,0) \succ_1 (1,3,2,0) =_1 (1,2,3,0) \succ_1 (1)\\
P^2 &: (2,3,0) \succ_2 (2,1,0) \succ_2 (2,1,3,0) \succ_2 (2,3,1,0) \succ_2 (2,0) \succ_2 (2)\\
P^3 &: (3,1,0) \succ_3 (3,1,2,0) \succ_3 (3,0) \succ_3 (3,2,0) \succ_3 (3,2,1,0) \succ_3 (3)
\end{aligned}
$$

Observe that node 2 had chosen path $(2, 3, 0)$ as the new best path. However, as only the edge $(2, 3)$, i.e., the edge to the next hop of the best path, has been included in the routing network, and node 3 has chosen $(3, 1, 0)$ as the new best path, adding the edge $(3, 1)$ to the routing network, the current available path of node 2 to destination 0 is $(2, 3, 1, 0)$ which is an alternative worse than the originally chosen one. Colored paths represent again the decision situation for node 2. The new configuration is

$$\beta_0(\pi) = 0, \quad \beta_1(\pi) = 0, \quad \beta_2(\pi) = 1, \quad \beta_3(\pi) = 1.$$

Given the network for $t = 1$, another simultaneous update step of all nodes gives the routing network for time step $t = 2$.



$$
\begin{aligned}
P^0 &: (0)\\
P^1 &: (1,0) \succ_1 (1,2,0) \succ_1 (1,3,0) \succ_1 (1,3,2,0) =_1 (1,2,3,0) \succ_1 (1)\\
P^2 &: (2,3,0) \succ_2 (2,1,0) \succ_2 (2,1,3,0) \succ_2 (2,3,1,0) \succ_2 (2,0) \succ_2 (2)\\
P^3 &: (3,1,0) \succ_3 (3,1,2,0) \succ_3 (3,0) \succ_3 (3,2,0) \succ_3 (3,2,1,0) \succ_3 (3)
\end{aligned}
$$

For $t > 2$, the configurations do not change anymore. We have reached an equilibrium or fixed point.

Proposition 1.1 *Let $G = (V, E)$ be any connected graph and let the path sets P^u be totally ordered for all $u \in V$. Let π be any configuration. Then, it holds*

$$\pi = \beta(\pi) \iff \pi \text{ is stably confluent}$$

Proof: We prove both directions individually.

($\Leftarrow$) By definition.

($\Rightarrow$) Suppose $\pi = \beta(\pi)$. It is enough to show that $\beta_u(\pi) \neq u$ for all $u \in V \setminus \{0\}$, as in this case each node $u \in V \setminus \{0\}$ has exactly one path to 0. Assume to the contrary, $\beta_u(\pi) = u$ for some $u \in V \setminus \{0\}$. Since G is connected, there is such a u so that $N(u)$ contains a node v possessing a path p to 0. Thus, by prolonging path p with u as the new first node we obtain $(u, p) \succ_u (u)$. A contradiction.

This completes the proof of the proposition. ■

1.3 Fixed-point analysis

A fundamental question in Interdomain routing is: When does BGP converge?

If we investigate this question algorithmically then we ask if a given BGP system has a fixed point or allows for reaching a fixed point. Here, a BGP system is defined to consist of a graph $G = (V, E)$ and a family of ranked path-set in G to destination $0 \in V$ (in the above-mentioned sense) where we assume that not all paths to 0 need be included.

Answers to these questions are negative, in general:

- There is no guarantee of convergence (see examples on the Assignments).
- It is NP-complete to decide if a BGP system has a stably confluent configuration [18].

- It is PSPACE-complete to decide if a BGP system always converges into some stably confluent state [8].

Why, then, does BGP appear to be rather stable in everyday experience?

There are rational rules for local policies which ensure convergence, but they are not part of the protocol specification. These rules are based on contractual business relationships between Autonomous Systems which typically belong to a certain Internet Service Provider. For simplification, let us assume that there are only two types of Autonomous Systems: *customers* and *providers*. Customers buy routes from providers to get global connectivity to 0. Accordingly, providers sell routs to customers.

Given a graph $G = (V, E)$ we can decompose the neighborhood of a node $v \in V$:

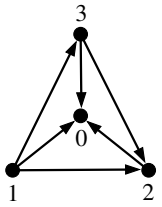
- $\mathrm{Cust}(v)$ is the set of all customers of v (buying routes from v)
- $\mathrm{Prov}(v)$ is the set of all providers of v (selling routes to v)

The graph G can be oriented given such a decmposition. Let $u, v \in V$ be two nodes such $\{u, v\} \in E$. Then,

- an oriented edge $u \to v$ indicates that $u \in \mathrm{Cust}(v)$ and
- an oriented edge $u \leftarrow v$ indicates that $u \in \mathrm{Prov}(v)$.

The orientation can be extended to paths. A path p in G is said to be *valley-free* if and only if its orientation pattern belongs to $\to^*\leftarrow^*$.

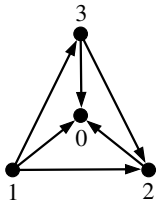
Example: We consider a graph with contracts among the nodes:



- The path $(1, 3, 2, 0)$ is valley-free: $\to \to \to$
- The path $(3, 1, 2, 0)$ is not valley-free: $\leftarrow \to \to$
- The path $(1, 3, 0, 2)$ is valley-free: $\to \to \leftarrow$

A path $p = (u_0, u_1, \ldots, u_m)$ (with $u_m = 0$) is called a *customer route* iff $u_1 \in \mathrm{Cust}(u_0)$, and is called a *provider route* iff $u_1 \in \mathrm{Prov}(u_0)$.

Example (cont'd): We classify routes other than ε for the graph above. Red routes are customer routes, black routes are provider routes.



$$
\begin{aligned}
P^1 &: (1,0) \succ_1 (1,2,0) \succ_1 (1,3,0) \succ_1 (1,3,2,0) =_1 (1,2,3,0) \\
P^2 &: (2,3,0) \succ_2 (2,1,0) \succ_2 (2,1,3,0) \succ_2 (2,3,1,0) \succ_2 (2,0) \\
P^3 &: (3,1,0) \succ_3 (3,1,2,0) \succ_3 (3,0) \succ_3 (3,2,0) \succ_3 (3,2,1,0)
\end{aligned}
$$

Given these notions we can formulate the Gao-Rexford convergence criterion [13]: A BGP system converges into a stable confluent state if the following three conditions are fulfilled:

1. All paths are valley-free.
2. The oriented graph is acyclic.
3. For all nodes a and all paths p, q beginning with a, if p is a customer route and q is a provider route then $p \succ_a q$.

Note that this criterion expresses only sufficient conditions for convergence.

For instance, if we eliminate all routes from the example above that are not valley-free then the Gao-Rexford criterion applies to the BGP system.

Networks 2

We foster a data-driven approach to understanding dynamical network behavior.

2.1 Network exploration and analysis

Sketch of research pipeline in empirical sciences:

1. Theory ← often omitted from the cycle
2. Hypotheses
3. Research design
4. Data collection
5. Exploration and analysis ← often data-driven studies start here
6. Interpretation and presentation

These steps are iterated (when new evidence comes in).

Step 5 will be the focus of this course. We will study

- formal and algorithmic concepts
- simulation and modelling techniques

for evaluting time-dependent network data.

2.2 Network data

2.2.1 Data

Data refers to variables for *entites* (or *units of observation*). More specifically,

- A is a set of (atomic) *items*,
- for $i \in A$, variable x_i represents values of a common *attribute* for all items in A, i.e., x is a mapping $x : A \to R : i \mapsto x_i$, or $x = (x_i)_{i \in A}$ where $x_i \in R$,

- R is the *range* of x, A is called the *domain* of x

Typically in emprical research, multiple attributes are collected in tables where the columns represent items and the rows represent attributes.

According to the range, attributes can be classified:

- *nominal* or *categorical*: there are no relationships among the elements of the range other than equality or inequality (e.g., names, types, labels)
- *ordinal*: the range satisfy certain order properties such as required for weak orders, preference relations, rankings (e.g., paths in policy routing)
- *numerical*: the range consists of number such as $\mathbb{N}$ or $\mathbb{R}_{\geq 0}$.

We assume that 0 represents a massing or neutral datum.

2.2.2 Dyadic data

Entities need not be atomic; they can be compound objects of more elementary entities. A *dyad* is a pair of items.

> **Example:** In a study we could explore relationship among married couples. The relevant data may include:
>
> - attributes of individuals: gender, income, personality
> - attributes of the couple: age difference, duration of marriage, number of children

The general assumption in (classical) dyadic data analysis is that dyads are independent.

We say that two dyads *overlap* if and only if they share a member. This gives us the characteristic of network data:

1. Units of observation are dyads.
2. Dyads are overlapping.

That is, the essential assumption in network analysis is that dyads are dependent. We even cab define network analysis as the study of effects of overlapping dyads.

2.2.3 Time-dependent data

Attribute values may change over time. And, there are differences in how data can depend on time. In general, data time-dependent data can be classified as follows:

- *panel data* (or *longitudinal data*): we have attributes values of all items for at least two points in time, i.e., $x(1), x(2), \ldots, x(k)$ where $x(j) = (x_i(j))_{i \in A}$.
- *time-series data*: we have attribute values of a single item over time.
- *cross-sectional data*: we have attribute values of all items for one specific point in time.
- *event data*: we have attribute values for items labelled with a time stamp (e.g., log files, audit trails, live scores, etc.)

Typically, event data are transformed into panel data.

2.3 Network representations

We adopt a network view where we consider networks to be representations of a specific format. That is, we are not so much interested in *what* is represented, but *how* it is represented.

2.3.1 Whole networks

As overlapping dyads are the fundamental objects of network analysis, we need a notion to collect all possible dependencies among dyads. This is done by introducing interaction domains.

Definition 2.1 *Let A be a set of items. An* interaction domain *$\mathcal{I}$ on A is a binary, symmetric relation $\mathcal{I} \subseteq A \times A$.*

In many cases, $\mathcal{I} = A \times A$ or $\mathcal{I} = (A \times A) \setminus \{ (i, i) \mid i \in A \}$. However, when studying BGP systems, the interaction domain is the AS graph.

Definition 2.2 *Let A be a set of items. A* (whole) network *consists of a set of attributes on an interaction domain $\mathcal{I} \subseteq A \times A$ and a (possibly empty) set of attributes on A.*

For a network, items of A represent *actors*, and attribute values $x_{i,j} \neq 0$, where $(i,j) \in \mathcal{I}$, are *ties*. Notice that x_{ij} is a usual abbreviation for $x_{(i,j)}$ for any dyad $(i,j) \in \mathcal{I}$.

Definition 2.3 *Let $x : \mathcal{I} \to R$ be an attribute defined on an interaction domain $\mathcal{I} \subseteq A \times A$. The (weighted, directed)* graph $G(x) = (V, E, w)$ of network x *consists of*

- *vertex set* $V =_{\text{def}} A$,
- *edge set* $E =_{\text{def}} \{\ (i,j) \in \mathcal{I} \mid x_{ij} \neq 0\ \}$, *and*
- *edge weights* $w : E \to R : (i,j) \mapsto x_{ij}$.

If $x_{ij} = x_{ji}$ for all $(i,j) \in \mathcal{I}$, $G(x)$ can be defined correspondingly as an undirected graph.

A completion of an attribute to the full interaction domain $A \times A$ by imputing zeroes gives the *adjacency matrix* of the associated weighted graph, which is another representation of a network.

Definition 2.4 *Let $x : \mathcal{I} \to R$ be an attribute defined on an interaction domain $\mathcal{I} \subseteq A \times A$. The (binary)* relation $\to \subseteq A \times A$ of network x *is defined by*

$$(i,j) \in \to \quad \Longleftrightarrow_{\text{def}} \quad (i,j) \in \mathcal{I} \wedge x_{ij} \neq 0$$

In infix notation, this is written as $i \to j$.

2.3.2 Two-mode networks

Assume that the observational units are relations between pairs of items of different types, e.g., users and fan sites on Facebook, authors and scientific papers, or politicians and boards.

We generalize interaction domains.

Definition 2.5 *An* affiliation domain *is a relation $\mathcal{A} \subseteq A \times S$ on disjoint sets A and S.*

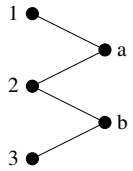
Definition 2.6 *A* two-mode network *consists of a set of attributes on an affiliation domain $\mathcal{A} \subseteq A \times S$ and a (possibly empty) set of attributes on A and S.*

All notions for networks translate to two-mode networks. Note that two-mode networks are bipartite by definition.

Definition 2.7 *Let $X \in R^{n \times m}$ be the matrix associated with a two-mode network attribute, $\|A\| = n$ and $\|S\| = m$. The networks associated with the matrices $X \cdot X^T$ and $X^T \cdot X$ are called* one-mode projections.

Note that the interaction domain of $X \cdot X^T$ is $A \times A$ and the interaction domain of $X^T \cdot X$ is $S \times S$.

Example: Consider sets $A = \{1, 2, 3\}$ and $S = \{a, b\}$. Suppose a two-mode network attribute is given by the following graph and the associated matrices X and X^T:

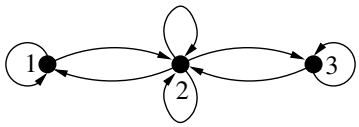


$$X = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad X^T = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Then, we calculate

$$X \cdot X^T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

The (multi)graph of the network can be drawn as follows:



Analogously, we calculate for the other one-mode projection

$$X \cdot X^T = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

The (multi)graph of the network can be drawn as follows:

Note that each directed edge $u \to v$ respresents one walk from u to v.

2.3.3 Ego and personal networks*

Definition 2.8 *An affiliation domain $\mathcal{A} \subseteq A \times S$ is said to be* egocentric *if and only if $\|\{\ s \mid (i, s) \in \mathcal{A}\ \}\| = 1$ for all $i \in A$. In other words, every element of A is affiliated with exactly one element of S.*

In an egocentric domain, elements of S are called *egos*, and elements of A are called *alteri*. An egocentric domains is uniquely decomposable into its *ego partition* (see example below).

Definition 2.9 *Given a two-mode network on an egocentric domain, each restriction of its attributes to an element of the ego partition defines an* ego network.

Definition 2.10 *Let A and S be disjoint sets, let $\mathcal{I} \subseteq A \times A$ be an interaction domain, and let $\mathcal{A} \subseteq A \times S$ be an egocentric affiliation domain. For a set of attributes defined on $\mathcal{I}$ and $\mathcal{A}$, every restriction induced by an element of the ego partition defines one* personal network.

Example: ...

2.3.4 Time-dependent networks

We consider attributes on an interaction (or affiliation) domain changing over time. The focus is on panel network-data.

Definition 2.11 *A* time-dependent network *is a set of attributes on an interaction domain $\mathcal{I} \subseteq A \times A$ and a (possibly empty) set of attributes on A, where all atributes depend on (same) time $t \in \mathbb{N}$.*

Note that we consider time-discrete networks.

Iterated network maps 3

3.1 Networks as dynamical systems

In this section, we want to introduce specific formal notions for studying the dynamical behavior of networks. We restrict ourselves to single-attribute networks (with a fixed interaction domain).

Let $x : \mathcal{I} \to R$ be an attribute. For the sake of convenience, we assume that x is a numerical attribute. Furthermore, we consider an infinite sequence of identical copies of x, i.e., $(x(t))_{t\in\mathbb{N}}$ or $x : \mathcal{I} \times \mathbb{N} \to R$. The attribute values are called *states*. The set of all possible sequences is called a *process*; one specific sequence is called *trajectory*. A *dynamic* F is a mechanism for selecting trajectories of a process. A dynamic makes assumptions on how the state at time step k will look like; here, depending only on the initial state z_0 and time k. We thus can express a dynamic as a sequence $(\varphi_t)_{t\in\mathbb{N}}$ where $\varphi_k : R^I \to R^I$.

We adopt notions and notations from dynamical systems. That is, the functions φ_k are iterated maps. Let $F : R^I \to R^I$ be any function. Then, inductively define

$$F^0(z) =_{\text{def}} z, \qquad F^k(z) =_{\text{def}} F(F^{k-1}(z)) \text{ for } k > 0.$$

So, $\varphi_k = F^k$.

The following summarizes the notions schematically:

$$\begin{array}{ccccccccccccc}
x(0) & \to & x(1) & \to & x(2) & \to & \dots & \to & x(k) & \to & \dots & & \text{process} \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow & & & & \\
z_0 & \to & z_1 & \to & z_2 & \to & \dots & \to & z_k & \to & \dots & & \text{trajectory} \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow & & & & \\
\varphi_0(z_0) & \to & \varphi_1(z_0) & \to & \varphi_2(z_0) & \to & \dots & \to & \varphi_k(z_0) & \to & \dots & & \text{dynamic} \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow & & & & \\
F^0(z_0) & \to & F^1(z_0) & \to & F^2(z_0) & \to & \dots & \to & F^k(z_0) & \to & \dots & & \text{iterated map}
\end{array}$$

Notice that iterated maps describe memory-less dynamics. In this sense, they are deterministic versions of Markov chains.

3.2 The phase space

We investigate dynamics induced by iterating a map $F : D^n \to D^n$ where, more commonly, D denotes the domain of F (i.e., D corresponds to the range of an attribute) and n is the number of items or dyads.

A fundamental concept in the study of iterated maps is the orbit.

Definition 3.1 *Let $F : D^n \to D^n$ be a total mapping. Then, the* orbit of z_0 under F *is defined to be the sequence $(z_0, z_1, z_2, \ldots, z_k, \ldots)$ such that $z_k = F^k(z_0)$ for all $k \in \mathbb{N}$.*

An orbit is a specific trajectory.

Example: We discuss some examples of iterated maps and orbits.

- Let $D = \mathbb{N}$, $n = 1$, and $F(x) =_{\text{def}} x + 1$. The orbit of 0 is $(0, 1, 2, 3, \ldots)$.
- Let $D = \mathbb{R}_{\geq 0}$, $n = 1$, and $F(x) =_{\text{def}} \frac{x}{x+1}$. The orbit of 1 is $(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots)$.
- A prominent iterated map is the Bernoulli shift: Let $D = [0, 1)$, $n = 1$, and $F : D \to D : x \mapsto 2x \bmod 1$, i.e.,

$$F(x) =_{\text{def}} \begin{cases} 2x & \text{if } 0 \leq x < \frac{1}{2} \\ 2x - 1 & \text{if } \frac{1}{2} \leq x < 1 \end{cases}$$

  Suppose $x \in [0, 1)$ is given in binary expansion as $0.a_1a_2a_2\ldots$, $a_i \in \{0, 1\}$, i.e., it holds that $x = \sum_{i=1}^{\infty} a_i 2^{-i}$. Without loss of generality, and to avoid ambuigity, we only consider binary expanions with an infinite number of zeroes. We have two cases:

  - *Case* $a_1 = 0$: That is, $x < \frac{1}{2}$. Applying F to x gives

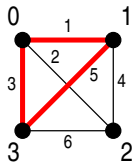
$$\begin{aligned} F\left(\sum_{i=1}^{\infty} a_i 2^{-i}\right) = 2\sum_{i=1}^{\infty} a_i 2^{-i} &= \sum_{i=1}^{\infty} a_i 2^{-(i-1)} \\ &= \sum_{i=2}^{\infty} a_i 2^{-(i-1)} = \sum_{i=1}^{\infty} a_{i+1} 2^{-i} \end{aligned}$$

  - *Case* $a_1 = 1$: That is, $x \geq \frac{1}{2}$. Analogously to the first case, applying F to x gives

$$\begin{aligned} F\left(\sum_{i=1}^{\infty} a_i 2^{-i}\right) = 2\sum_{i=1}^{\infty} a_i 2^{-i} - 1 &= \sum_{i=1}^{\infty} a_i 2^{-(i-1)} - 1 \\ &= 1 + \sum_{i=2}^{\infty} a_i 2^{-(i-1)} - 1 \\ &= \sum_{i=2}^{\infty} a_i 2^{-(i-1)} = \sum_{i=1}^{\infty} a_{i+1} 2^{-i} \end{aligned}$$

  Hence, $F(0.a_1a_2a_3\ldots) = 0.a_2a_3a_4\ldots$. The orbit of $x_0 = 0.a_1a_2a_3\ldots$ is the sequence $(x_k)_{k\in\mathbb{N}}$ such that $x_k = 0.a_{k+1}a_{k+2}a_{k+3}\ldots$.

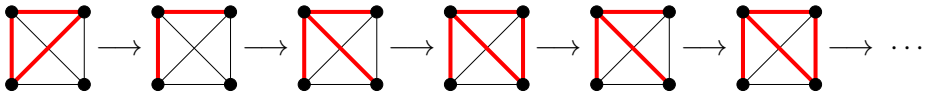
- Let $D = \{0, 1\}, n = \|\{\ (i,j) \in \mathcal{I} \mid i < j\ \}\|$, and $\mathcal{I} = A \times A \setminus \{\ (i,i) \mid \in A\ \}$ for some set A be given. That is, we consider an interaction domain representing a complete undirected graph. So, n is the number of edges, i.e., $n = \binom{\|A\|}{2}$. Suppose edges are lexicographically enumerated. Then, $x \in D^n$ encodes an undirected graph. For example, the following graph represents the interaction domain for $\|A\| = 4$ and edges colored red represent the value $x_{ij} = 1$:



Define $F : D^n \to D^n$ to be the mapping that satisfies

$$F(x)_{\max\{k \mid x_k = 1\}} = 0 \qquad \text{if } x \text{ contains a cycle}$$
$$F(x)_{\min\{k \mid x_k = 0\}} = 1 \qquad \text{if } x \text{ contains no cycle}$$

with all other components of $F(x)$ unchanged compared to x. Then, an example of an orbit for $\|A\| = 4$ are the following:



Proposition 3.2 *Let $F : D^n \to D^n$ be a total mapping, and let $x, y \in D^n$. Then, the orbits of x and y under F are either disjoint or there exist $k \in \mathbb{N}$ and $r \in \mathbb{Z}$ such that $F^{k'}(x) = F^{k'+r}(y)$ for all $k' \geq k$.*

Proof: Suppose the orbits of x and y are not disjoint, i.e., there are $t, t' \in \mathbb{N}$ such that $F^t(x) = F^{t'}(y)$. Define $r =_{\text{def}} t' - t$. So, $t' = t + r$. Then, by induction on $\ell \in \mathbb{N}$, we obtain that $F^{t+\ell}(x) = F^{t+r+\ell}(y)$ for all $\ell \in \mathbb{N}$:

- *Base of induction* $\ell = 0$: Then, $F^{t+0}(x) = F^t(x) = F^{t'}(y) = F^{t+r+0}(y)$.
- *Inductive step* $\ell > 0$: By the induction assumption we conclude that

$$F^{t+\ell}(x) = F(F^{t+\ell-1}(x)) = F(F^{t+r+\ell-1}(y)) = F^{t+r+\ell}(y).$$

Hence, setting $k =_{\text{def}} t$ and $k' =_{\text{def}} t + \ell$ proves the proposition. ■

Given a map $F : D^n \to D^n$, all orbits under F are collected in the phase space. The fundamental problem (in statistical mechanics) is getting knowledge on the probability distribution over the phase space, i.e., to determine the visiting probablity of a certain state in an orbit.

The following concepts are essential for addressing this question.

Definition 3.3 *Let $F : D^n \to D^n$ be a total mapping.*

1. *A state $x \in D^n$ is called* fixed point of F *if and only if $F(x) = x$.*
2. *A state $x \in D^n$ is called* periodic under F *if and only if there exists a $k \in \mathbb{N}_+$ such that $F^k(x) = x$. The number $k_0 \in \mathbb{N}_+$ minimal subject to $F^{k_0}(x) = x$ is called the* periodic order *of x, and x is then called* periodic of order k_0.
3. *A state $x \in D^n$ is called* transient under F *if and only if $F^k(x) \neq x$ or all $k \in \mathbb{N}_+$, i.e., x is not periodic.*

Obviously, a *fixed point* is a periodic state of order 1.

Example: We give examples for each part of the definition.

- For the Bernoulli shift, 0 is the only fixed point, $\frac{2}{3}$ is an example of a periodic state of order 2, and $\frac{1}{\sqrt{2}}$ is an example of a transient state.
- Consider again the map $F : \{0,1\}^n \to \{0,1\}^n$ from above for $n = 4$. Then, there is no fixed point of F, ... are periodic of order 2, and ... is transient.

The following proposition explains why recurring states are referred to as "periodic."

Proposition 3.4 *Let $F : D^n \to D^n$ be a total function. Let $x \in D^n$ be a periodic state of order k_0, and let $k \in \mathbb{N}$. Then, the following holds:*

$$F^k(x) = x \iff k_0 \text{ divides } k$$

Proof: We prove both directions individually.

($\Leftarrow$) Observe that $x = F^{k_0}(x) = F^{k_0}(F^{k_0}(x))$. An easy inductive argument shows that $x = F^{c \cdot k_0}(x)$ for all $c \in \mathbb{N}$. Hence, if k_0 divides k, i.e., $k = c \cdot k_0$ for some $c \in \mathbb{N}$, then $F^k(x) = x$.

($\Rightarrow$) Case $k = 0$ is trivial. Now, suppose $k \geq k_0 > 0$. Then, $k = c \cdot k_0 + r$ for uniquely determined $c \in \mathbb{N}_+$ and $r \in \{0, 1, \ldots, k_0 - 1\}$. Thus,

$$x = F^k(x) = F^{c \cdot k_0 + r}(x) = F^r(F^{c \cdot k_0}(x)) = F^r(x)$$

Since k_0 is the smallest positive number with this property, it follows that $r = 0$. Hence, $k = c \cdot k_0$. So, k_0 divides k.

This proves the proposition. ■

Definition 3.5 *Let $F : D^n \to D^n$ be a total mapping. Let x be a periodic state of order k. Then, the set $\{x, F(x), F^2(x), \ldots, F^{k-1}(x)\}$ is called a* limit cycle *(of length k) of F.*

Limit cycles (of length k) are also called *attractors* (of length k). The limit cycle corresponding to a fixed point is also called *singleton attractor*.

Any easy consequence of Proposition 3.2 is that limit cycles are either disjoint or identical.

Corollary 3.6 *Let $F : D^n \to D^n$ be a map. If $\{x_1, \ldots, x_\ell\}$ and $\{y_1, \ldots, y_r\}$ are two limit cycles of F such that $\{x_1, \ldots, x_\ell\} \cap \{y_1, \ldots, y_r\} \neq \emptyset$ then $\{x_1, \ldots, x_\ell\} = \{y_1, \ldots, y_r\}$.*

For finite domains, orbits have a simple structure.

Proposition 3.7 *Let $F : D^n \to D^n$ be a map over a finite domain D. Let $(x_i)_{i\in\mathbb{N}}$ be the orbit of $x_0 \in D^n$ under F. Then, there are $k_0 \in \mathbb{N}$ and $\ell_0 \in \mathbb{N}_+$ such that*

(a) $\{x_0, \ldots, x_{k_0-1}\}$ is the set of k_0 transient states of the orbit of x_0 under F and

(b) $\{x_{k_0}, \ldots, x_{k_0+\ell_0-1}\}$ is a limit cycle of length ℓ_0 of F.

Proof: Let $(x_i)_{i\in\mathbb{N}}$ be the orbit of $x_0 \in D^n$ under F, i.e., $x_i = F^i(x_0)$. D is finite, so is D^n. Thus, there are $k \geq 0$ and $\ell > 0$ such that $F^k(x_0) = x_k = x_{k+\ell})F^{k+\ell}(x_0)$. Define parameters k_0 and ℓ_0 as follows (in this order):

$$\begin{aligned} k_0 &=_{\text{def}} \min \{\ k \mid F^k(x_0) = F^{k+r}(x_0) \text{ for some } r > 0\ \} \\ \ell_0 &=_{\text{def}} \min \{\ r \mid F^k(x_0) = F^{k_0+r}(x_0)\ \} \end{aligned}$$

Then, for all $r > 0$, it holds that

$$F^{k_0+r}(x_0) = F^r\left(F^{k_0}(x_0)\right) = F^r\left(F^{k_0+\ell_0}(x_0)\right) = F^{k_0+\ell_0+r}(x_0).$$

Hence, x_i is periodic of order ℓ_0 if $i \geq k_0$, which is tatement (b), and x_i is transient if $i < k_0$, which is statement (a). This proves the Proposition. ■

The orbit under a iterated, finite-domain map can be visualized by the following transition diagramm:

...

In principle, iterated maps can be studied graph-theoretically. A map $F : D^n \to D^n$ over a finite domain D can be associated with the directed graph $\Gamma(F) = (V, E)$, called *state graph of* F, where

$$V =_{\text{def}} D^n, \qquad E =_{\text{def}} \{\ (x, F(x)) \mid x \in D^n\ \}.$$

Note that $\Gamma(F)$ might have loops.

According to Proposition 3.2, Corollary **??**, and Proposition **??**, the state graph of F can be uniquely decomposed into

- disjoint cycles $C_1, \ldots, C_k$ (representing limit cycles) and
- disjoint (directed) trees $T_1, \ldots, T_r$ (representing transient states) each of which is incident with exactly one cycle $C_1, \ldots C_k$

Example: We consider the map

$$F : \{0,1\}^3 \to \{0,1\}^3 : (x_1, x_2, x_3) \mapsto (x_2 \oplus x_3, 1 \oplus x_1 \oplus x_3, x_! \oplus x_2)$$

where $\oplus$ denotes XOR or, equivalently, addition modulo 2. In order to determine the state graph of F, we first represent F as a truth table:

(x_1, x_2, x_3)	$F(x_1, x_2, x_3)$
000	010
001	100
010	111
011	001
100	001
101	111
110	100
111	010

From this, we easily obtain the state graph of F:

...

A cycle together with all its incident trees is called *basin of attraction*.

It is clear that transient states have visiting probability zero. The following proposition gives the precise visiting probability of a periodic state in terms of the structure of its corresponding basin of attraction.

Proposition 3.8 *Let $F : D^n \to D^n$ be a map over a finite domain D, $\|D\| = m$. Let $z \in D^n$ be periodic, and let $E \subseteq D^n$ be the basin of attraction of (the limit cycle of) z. Suppose E consists of s transient and r periodic states. Then, the visiting probability of z in a random orbit is*

$$\left(1 + \frac{s}{r}\right) \cdot m^{-n}.$$

Proof: Let $x \in D^n$ be an arbitrary state. Consider the orbit $(x_i)_{i\in\mathbb{N}}$ such that $x = x_0$ and $F^k(x_0) = x_k$ for all $k > 0$. Suppose $x_0, \ldots, x_{k_0-1}$ are all transient states and $x_{k_0}, \ldots, x_{k_0+r-1}$ are all periodic states (of order r). Let $z \in D^n$ be a state in the orbit $(x_i)_{i\in\mathbb{N}}$. Define

$$P_z =_{\text{def}} \mathbf{P}\left[z \text{ is visited in } (x_i)_{i\in\mathbb{N}}\right].$$

Then, P_z is given by a frequency sequence of the initial segments of the orbit:

$$P_z = \lim_{N\to\infty} \frac{\|\{\ i \mid i \in \{0,1,\ldots,N-1\} \text{ and } x_i = z\ \}\|}{N}$$

To calculate P_z, we have two cases.

- Suppose z is transient. Thus, $\|\{\ i \mid i \in \{0,1,\ldots,N-1\} \text{ and } x_i = t\ \}\| = 1$ for $N \geq k_0$. Hence,
$$P_z = \lim_{N\to\infty} \frac{1}{N} = 0.$$

- Suppose z is perdiodic. Thus, for $N \geq k_0 + 1$,
$$\left\lfloor \frac{N-k_0-1}{r} \right\rfloor \leq \|\{\ i \mid i \in \{0,1,\ldots,N-1\} \text{ and } x_i = t\ \}\| \leq \left\lfloor \frac{N-k_0-1}{r} \right\rfloor + 1.$$
Hence, we obtain
$$\begin{aligned} P_z &\leq \lim_{N\to\infty} \frac{\frac{N-k_0-1}{r}+1}{N} = \lim_{N\to\infty} \frac{N-k_0-1+r-1}{rN} = \frac{1}{r} \\ P_z &\geq \lim_{N\to\infty} \frac{1+\frac{N-k_0-1}{r}-1}{N} = \lim_{N\to\infty} \frac{N-k_0-1-r}{rN} = \frac{1}{r} \end{aligned}$$
Consequently, $P_z = \frac{1}{r}$.

Now, consider any periodic $z \in D^n$ following the specification given in the proposition. Then, z lies on $s + r$ orbits. So, the visiting probability of z is

$$\mathbf{P}[z \text{ is visited in some orbit}] = \frac{s+r}{m^n} \cdot \frac{1}{r} = \left(1 + \frac{s}{r}\right) \cdot m^{-n}.$$

This proves the proposition. ■

Example (cont'd): Consider the map $F : \{0,1\}^3 \to \{0,1\}^3$ from above. The visiting probabilities for the periodic states $010, 111, 001, 100$ are all $\frac{1}{4}$.

3.3 Series, levels, and plots*

Even for finite domains D, the state graph $\Gamma(F)$ for an iterated map $F : D^n \to D^n$ can be too large to construct explicitly. Since $\|V\| = \|E\| = \|D\|^n$, it has size exponential in the number n. If we consider boolean network attributes then the size $2^{O(n^2)}$. This forces us to reduce the dimensionality of the phase space. In this subsection, we look at three methods to achieve this reduction.

(Multivariate) Time series

Let $F : D^n \to D^n$ be a map. Let $\tau : D^n \to \mathbb{R}$ be any function. Then, the *time series* $(\tau_i)_{i\in\mathbb{N}}$ associated with an orbit $(x_i)_{i\in N}$ is given by

$$\tau_i = \tau(x_i) = \tau(F^i(x_0)).$$

The following summarizes the construction of a time series schematically:

$$\begin{array}{ccccccccccccc} x_0 & \stackrel{F}{\to} & x_1 & \stackrel{F}{\to} & x_2 & \stackrel{F}{\to} & \dots & \stackrel{F}{\to} & x_k & \stackrel{F}{\to} & \dots & & \text{orbit} \\ \tau\downarrow & & \tau\downarrow & & \tau\downarrow & & & & \tau\downarrow & & & & \\ \tau_0 & \to & \tau_1 & \to & \tau_2 & \to & \dots & \to & \tau_k & \to & \dots & & \text{time series} \end{array}$$

If $\tau : D^n \to \mathbb{R}^m$ has m-dimensional function values then the sequence $(\tau_i)_{i\in\mathbb{N}}$ is called *multivariate* (or *multidimensional*) *time series*.

> **Example:** A typical example of derived time series from an underlying dynamics is the evolution of the voting distribution among voters over some time period. (We will study this in the opinion dyamics section.)
>
> For a more technical example, consider the following state graph:
>
> ...
>
> Several time series can be derived from the orbit of this state graph.
>
> - Define $\tau_1 : \{0,1\}^4 \to \mathbb{R} : (x_1, x_2, x_3, x_4) \mapsto |(x_1, x_2, x_3, x_4)|_1$ (where $|x|_a$ denotes the number of a's in a tuple $x \in \{0,1\}^n$)
> - For a bivariate times series, define
>
> $$\tau_2 : \{0,1\}^4 \to \mathbb{R}^2 : (x_1, x_2, x_3, x_4) \mapsto (|(x_1, x_2, x_3, x_4)|_0, |(x_1, x_2, x_3, x_4)|_1)$$
>
> - Define $\tau_3 : \{0,1\}^4 \to \mathbb{R} : (x_1, x_2, x_3, x_4) \mapsto x_1 \oplus x_2 \oplus x_3 \oplus x_4$ (i.e., τ_3 is a parity function)
> - Define $\tau_3 : \{0,1\}^4 \to \mathbb{R} : (x_1, x_2, x_3, x_4) \mapsto x_2$ (i.e., τ_4 is a projection function)

Levels

Let $F : D^n \to D^n$ be a map. A subset $L \subseteq \{1, \ldots, n\}$ of size $\|L\| = m$ is called *level of* F if and only if $L = \emptyset$ or there is a map $G : D^m \to D^m$ such that for all $x \in D^n$,

$$\pi(F(x)) = G(\pi(x)),$$

where $\pi : D^n \to D^m$ is the projection that maps $(x_1, \ldots, x_n)$ to those m components indexed by the set L, i.e., $\pi(x_1, \ldots, x_n) = (x_{i_1}, \ldots, x_{i_m})$ and $L = \{i_1, \ldots, i_m\}$.

Note that by an inductive argument, it follows that:

$$\pi(F^k(x)) = G^k(\pi(x)) \quad \text{for all } k \in \mathbb{N}$$

Indeed, case $k = 0$ is trivial, and for the case $k > 0$, we obtain by using the inductive assumption

$$\pi(F^k(x)) = \pi(F(F^{k-1}(x)) = G(\pi(F^{k-1}(x))) = G(G^{k-1}(\pi(x))) = G^k(\pi(x)).$$

So, the iterated map G induces a subdynamic on the elements of the level L.

This can be summarized schematically as follows:

$$\begin{array}{ccccccccccc} x_0 & \stackrel{F}{\to} & x_1 & \stackrel{F}{\to} & x_2 & \stackrel{F}{\to} & \ldots & \stackrel{F}{\to} & x_k & \stackrel{F}{\to} & \ldots \quad \text{orbit} \\ \pi\downarrow & & \pi\downarrow & & \pi\downarrow & & & & \pi\downarrow & & \\ y_0 & \stackrel{G}{\to} & y_1 & \stackrel{G}{\to} & y_2 & \stackrel{G}{\to} & \ldots & \stackrel{G}{\to} & y_k & \stackrel{G}{\to} & \ldots \quad \text{orbit} \end{array}$$

Example: We determine all levels for the state graph $\Gamma(F)$ above:

- $\{2, 3, 4\}$ is a level of F with the map $G : \{0, 1\}^3 \to \{0, 1\}^3$
  ...
- $\{3, 4\}$ is a level of F (and of G) with the map $G' : \{0, 1\}^2 \to \{0, 1\}^2$
  ...

There are no other levels of F. As an example, consider the set $\{1, 3, 4\}$. The sequence $(\pi(F^i(.., .., ..)))_{i \in \mathbb{N}}$

The Derrida *plot*

This is a method for identifying turbulent behavior in the phase space.

Consider a map $F : \{0, 1\}^n \to \{0, 1\}^n$. Let $d_H(x, y)$ denote the Hamming distance between $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, i.e.,

$$d_H(x, y) =_{\text{def}} \|\{\ i \mid x_i \neq y_i\ \}\|.$$

The DERRIDA relation $\mathcal{D}(F)$ consists of the following multiset

$$\mathcal{D}(F) =_{\text{def}} \{\ (h_1, h_2) \quad | \quad \text{there are states } x, y \text{ such that } d_H(x,y) = h_1 \text{ and } d_H(F(x), F(y)) = h_2\ \}$$

The multiplicity of the pairs (h_1, h_2) is given by the number of pairs (x, y) that realize the values specified by h_1 and h_2.

Now, we can plot the relation $\mathcal{D}(F)$ as a diagram (with an appropriate representation of multiplicities).

> **Example:** We consider the following three maps for $n = 4$: Then, plot might look like as follows
>
> The curves are interpolations of resulting distances.

The intuition behind the DERRIDA plot is the following: The more pairs above the diagonal, the more chaos.

Using this intuition we can introduce a numerical measure for chaos. The measure is based on linear regression. Suppose $(x_1, y_1), \ldots, (x_N, y_N)$ are pairs in $\mathcal{D}(F)$ with multiple occurrences corresponding up to their multiplicities. We try to find a linear function βx which minimizes the distances to all pairs in the list, i.e., we want to find the right slope β. According to the methode of least squares we define

$$L(\beta) =_{\text{def}} \sum_{i=1}^{N} (y_i - \beta x_i)^2.$$

By taking derivatives, we obtain

$$L'(\beta) = \sum_{i=1}^{N} 2(y_i - \beta x_i)(-x_i), \qquad L''(\beta) = \sum_{i=1}^{N} 2x_i^2 > 0$$

Therefore, any zero of L' minimizes L. Hence, the optimale slope is

$$\beta = \frac{\sum_{i=1}^{N} y_i x_i}{\sum_{i=1}^{N} x_i^2}$$

The DERRIDA coefficient $\mathrm{Dc}(F)$ of a map is a scaled version of β:

$$\mathrm{Dc}(F) =_{\text{def}} \log_2 \beta$$

It is obvious that the DERRIDA coefficient ranges between $-\infty$ and ∞. The coefficient can be used to make distinction between different behaviors of iterated maps. The interpretations for F are as follows:

$$\begin{aligned} \mathrm{Dc}(F) \gg 0 &\quad : \quad F \text{ shows } \textit{chaotic} \text{ behavior} \\ \mathrm{Dc}(F) \approx 0 &\quad : \quad F \text{ shows } \textit{critical} \text{ behavior} \\ \mathrm{Dc}(F) \ll 0 &\quad : \quad F \text{ shows } \textit{frozen} \text{ behavior} \end{aligned}$$

The classification into these three types of behaviors becomes clear by looking at the following diagrams:

3.4 Local maps

An iterated map $F : D^n \to D^n$ describes a global behavior:

$$\begin{aligned} x_1 \leftarrow F(x_1, x_2, \ldots, x_n)_1 &= f_1(x_1, x_2, \ldots, x_n) \\ x_2 \leftarrow F(x_1, x_2, \ldots, x_n)_2 &= f_2(x_1, x_2, \ldots, x_n) \\ &\vdots \\ x_n \leftarrow F(x_1, x_2, \ldots, x_n)_n &= f_n(x_1, x_2, \ldots, x_n) \end{aligned}$$

global map local maps

In many cases, only local descriptions are available or even observable.

Let $f : D^n \to D$ be any function. A variable x_j (or an index j) is *fictive in* f if and only if

$$f(z_1, \ldots, z_{j-1}, z_j, z_{j+1}, \ldots z_n) = f(z_1, \ldots, z_{j-1}, z'_j, z_{j+1}, \ldots z_n)$$

for all $z_1, \ldots, z_{j-1}, z_{j+1}, \ldots z_n \in D$, $z_j, z'_j \in D$. Given a collection of functions (local maps) $f_1, \ldots, f_n : D^n \to D$, we say that x_i *depends on* x_j *in* f_i iff x_j is not fictive in f_i.

The *interdependence graph* of an iterated maps $F : \{0,1\}^n \to \{0,1\}^n$ (considered as the collection of its n local maps) is defined to consist of

- vertex set $V =_{\text{def}} \{1, \ldots, n\}$ and
- edge set $E = \{\ (i,j) \mid x_i \text{ depends on } x_j \text{ in } F\ \}$

We also uses undirected versions without loops.

Example: Let $F : \{0,1\}^3 \to \{0,1\}^3$ be given by the following truth table.

(x_1, x_2, x_3)	$F(x_1, x_2, x_3)$
000	100
001	101
010	011
011	011
100	100
101	101
110	101
111	101

Then, x_1 depends on x_1, x_2, x_2 depends on x_1, x_2, and x_3 depends on x_2, x_3.
So, the interpendence graph of F is as follows:

We briefly discuss the connection between the interdependence graph of an iterated network map and its underlying interaction domain. Suppose F is a map on an attribute x on a symmetrical interaction domain $\mathcal{I} \subseteq A \times A$.

1. Suppose x is an attribute on items, i.e., $x : A \to R$. Then, x_i depends on all x_j such that $\{i, j\} \in \mathcal{I}$. So, the (undirected) interdependence graph of F is a subgraph of $\mathcal{I}$.

2. Suppose x is an attribute on dyads, i.e., $x : \mathcal{I} \to R$. Then, x_{ij} depends on all x_e such that $e \in \mathcal{I}$ and $e \cap \{i, j\} \neq \emptyset$. So, the interdependence graph of F is a subgraph of $L(\mathcal{I})$, where $L(G)$ is the *line graph* of an undirected graph $G = (V, E)$, i.e., $L(G) = (V', E')$ such that

$$V' =_{\text{def}} E, \qquad E' =_{\text{def}} \{\ (e, f) \mid e, f \in E \text{ and } e \cap f = \emptyset\ \}$$

So far, we have considered a given iterated network map F decomposed into a collection of local network maps. We now turn our point of view and consider iterated network maps composed by a given collection of local network maps $f_1, \ldots, f_n$.

Suppose $x : \mathcal{I} \to R$ is a network attribute. Assume that $\mathcal{I}$ is enumerated $1, \ldots, n$. Let $M = \{\ f_i \mid i \in \mathcal{I}\ \}$ be a set of *local transitions* $f_i : D^{\deg_i + 1} \to D$ where $\deg_i$ denotes the in-degree of i in the interdependence graph. Additionally, suppose that we are given a mapping $\alpha : \{1, \ldots T\} \to \mathcal{P}(\mathcal{I})$ which is called *schedule*; the parameter $T > 0$ is any natural number.

For each $i \in \mathcal{I}$ and for each subset $U \subseteq \mathcal{I}$, *activity function* $\varphi_i[U]$ is defined for configuration $\vec{z} = (z_1, \ldots, z_n) \in D^n$ by

$$\varphi_i[U](\vec{z}) =_{\text{def}} \begin{cases} f_i(z_{i_1}, \ldots, z_{i_{\deg_i + 1}}) & \text{if } i \in U \\ z_i & \text{if } i \notin U \end{cases}$$

where $\{i_1, i_2, \ldots, i_{\deg_i + 1}\}$ is the set of neighbors in the interdependence graph.

For each set $U \subseteq \mathcal{I}$, the *global transition (function)* $\mathbf{F}_M[U] : D^n \to D^n$ is defined for configuration $\vec{z} = (z_1, \ldots, z_n)$ by

$$\mathbf{F}_F[U](\vec{z}) =_{\text{def}} \big(\varphi_1[U](\vec{z}), \ldots, \varphi_n[U](\vec{z})\big)$$

Finally, the *global network map* $\mathbf{F}_{(M,\alpha)} : D^n \to D^n$ *induced by* (M, α) is defined by

$$\mathbf{F}_{(M,\alpha)} =_{\text{def}} \prod_{k=1}^{T} \mathbf{F}_M[\alpha(k)],$$

i.e., $\mathbf{F}_{(M,\alpha)}$ is defined by the composition of global transition functions specified by the update schedule. Note that $f \cdot g$ is the function defined by $(f \cdot g)(x) = g(f(x))$. The

following shall elucidate the above definition in detail. For $T = 3$ and $\vec{z} \in D^n$, we have

$$\begin{aligned}
\mathbf{F}_{(M,\alpha)}(\vec{z}, 3) &= \left(\prod_{k=1}^{3} \mathbf{F}_M[\alpha(k)]\right)(\vec{z}) = \left(\mathbf{F}_M[\alpha(1)] \cdot \prod_{k=2}^{3} \mathbf{F}_M[\alpha(k)]\right)(\vec{z}) \\
&= \left(\prod_{k=2}^{3} \mathbf{F}_M[\alpha(k)]\right)\big(\mathbf{F}_M[\alpha(1)](\vec{z})\big) = \Big(\mathbf{F}_M[\alpha(2)] \cdot \mathbf{F}_M[\alpha(3)]\Big)\big(\mathbf{F}_M[\alpha(1)](\vec{z})\big) \\
&= \mathbf{F}_M[\alpha(3)]\big(\mathbf{F}_M[\alpha(2)]\big(\mathbf{F}_S[\alpha(1)](\vec{z})\big)\big)
\end{aligned}$$

Also notice that activity functions and global transitions do not depend on schedules.

Example: Suppose $\mathcal{I} = L(\mathcal{I}) = K^3$. Let $M = \{f_1, f_2, f_3\}$ consist of local transitions $f_i\{0,1\}^3 \to \{0,1\}$ such that for each $i \in \{1,2,3\}, z_1, z_2, z_3 \in \{0,1\}$

$$f_i \ : \ \{z_1, z_2, z_3\} \mapsto \begin{cases} z_i & \text{if } z_1 + z_2 + z_3 = 1 \\ 1 - z_i & \text{otherwise} \end{cases}$$

Let $U_1 = \{1,2\}$ and $U_2 = \{1,2,3\}$ be subsets of $\mathcal{I}$. Then we obtain the following activity functions:

- $\varphi_1[U_1] = f_1$, $\varphi_2[U_1] = f_2$, and $\varphi_3[U_1] = \mathrm{id}$;
- $\varphi_1[U_2] = f_1$, $\varphi_2[U_2] = f_2$, and $\varphi_3[U_2] = f_3$.

The global transition function looks as follows:

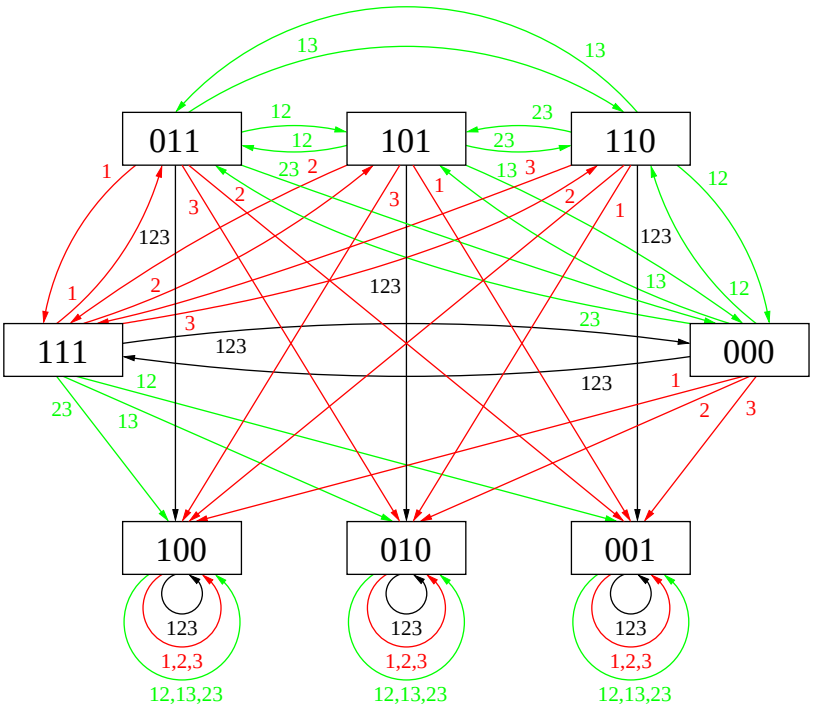
- $\mathbf{F}_M[U_1](z_1, z_2, z_3) = (f_1(z_1, z_2, z_3), f_2(z_1, z_2, z_3), z_3)$; concrete function values are, e.g.,

$$\begin{aligned}
\mathbf{F}_M[U_1](1,1,1) &= (0,0,1) \\
\mathbf{F}_M[U_1](1,0,1) &= (0,1,1) \\
\mathbf{F}_M[U_1](0,0,1) &= (0,0,1)
\end{aligned}$$

- $\mathbf{F}_M[U_2](z_1, z_2, z_3) = (f_1(z_1, z_2, z_3), f_2(z_1, z_2, z_3), f_3(z_1, z_2, z_3))$; concrete values are, e.g.,

$$\begin{aligned}
\mathbf{F}_M[U_2](1,1,1) &= (0,0,0) \\
\mathbf{F}_M[U_2](0,0,0) &= (1,1,1)
\end{aligned}$$

The following figure visualizes the global transition functions completely:



For the update schedules $\alpha_1 : \{1\} \to \mathcal{P}(\{1,2,3\})$ and $\alpha_2 : \{1,2,3\} \to \mathcal{P}(\{1,2,3\})$ given by

$$\alpha_1 : \{1\} \mapsto \{1,2,3\}, \qquad \alpha_2 : \begin{array}{l} \{1\} \mapsto \{2\} \\ \{2\} \mapsto \{1\} \\ \{3\} \mapsto \{3\} \end{array}$$

the induced network maps are as follows:

(x_1, x_2, x_3)	$F(x_1, x_2, x_3)$
000	111
001	001
010	010
011	100
100	100
101	010
110	001
111	000

(x_1, x_2, x_3)	$F(x_1, x_2, x_3)$
000	010
001	001
010	010
011	001
100	100
101	010
110	100
111	001

Agent-based modelling 4

Chapter to be worked out.

Agent-based modelling (ABM) is a (physics-oriented) methodology for studying complex networks. It uses simplified interaction models and simulations to explore a nonlinear dynamical behavior. ABM is typically applied when kinetic models involving differential/difference equation systems are inappropriate, e.g., due to the number and heterogeneity of variables.

The following fundamental design questions need to be answered when applying agent-based modelling:

- *Who has the agency?* – possible alternatives are actors or dyads.
- *What type of agency?* – possible alternatives are push or pull.
- *Which update orders?* – possible selection modes are deterministic or stochastic.

ABM is useful to gather information on the visiting probability distribution in the phase space; ideally, ABM runs converge to that distribution.

ABM replaces one application of an iterated map by a sequence of ministeps, chosen deterministically or randomly. This causes some problems, as ministeps introduce additional causality—causality understood as order of events. The effects of this additional causality are still unclear.

Deterministic simulation 5

We focus on, so called, *sequential dynamical systems* (SDS): A sequential dynamical system results from a dynamical system on n items or dyads by replacing each update step with a sequence of n update steps in which only one item or dyad is allowed to update its state. In this sense, the ministeps consist of permutations.

More formally, an update schedule $\alpha : \{1, \ldots, T\} \to \mathcal{P}(A)$ is called *sequential* if and only if $T = n$ and there is a permutation $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that $\alpha(i) = \{\pi(i)\}$. A sequential update schedule is usually identified with a permutation. An SDS $S = (G, L, \pi)$ consists of an interdependence graph $G = (V, E)$, without loss of generality $V = \{1, \ldots, n\}$, a set $L = \{f_1, \ldots, f_n\}$ of local transitions (compatible with G), and a permutation $\pi : V \to V$.

We study the effect of the choice of π on the phase space (state graph) of the induced global network map.

5.1 Functional equivalence

Definition 5.1 *Let L be a set of local transitions on V, $\|V\| = n$. Let $\pi, \pi' : X \to X$ be permutations on V. Then, π and π' are said to be* functionally equivalent, *$\pi \equiv_{\mathrm{f}} \pi'$ in symbols, if and only if $\mathbf{F}_{(L,\pi)} = \mathbf{F}_{(L,\pi')}$ for all $k \in \mathbb{N}_+$.*

A deeper analysis of the notion of functional equivalence is based on update orders given by permutations. Let S_V denote the symmetric group of V, i.e., the set all permutations $\pi : V \to V$. For distinct $\pi, \pi' \in S_V$, we say that π and π' are *adjacent* (with respect to a graph $G = (V, E)$) if and only if there is a k such that $\{\pi(k), \pi(k+1)\} \notin E$ and $\pi(i) = \pi'(i)$ for $i \notin \{k, k+1\}$. In other words, π and π' are adjacent with respect to G iff π' is obtained by swapping consecutive elements, not neighbored in G, in the permutation order of π.

Proposition 5.2 *Let $\pi, \pi' \in S_V$ be adjacent with respect to some $G = (V, E)$. Let $k \in V$ be such that $\{\pi(k), \pi(k+1)\} \notin E$ and $\pi(i) = \pi'(i)$ for all $i \notin \{k, k+1\}$. Then,*

$$\mathbf{F}_L[\pi(k)] \cdot \mathbf{F}_L[\pi(k+1)] = \mathbf{F}_L[\pi(k+1)] \cdot \mathbf{F}_L[\pi(k)]$$

for all sets L of local transition functions compatible with G.

Proof: Since $\{\pi(k), \pi(k+1)\} \notin E$, $\pi(k)$ is fictive in $f_{\pi(k+1)}$ and $\pi(k+1)$ is fictive in $f_{\pi(k)}$. That is, we can replace the $\pi(k)$-th argument in $f_{\pi(k+1)}$ as well as the $\pi(k+1)$-st

argument in $f_{\pi(k)}$ arbitrarily. Suppose L is a set of local transitions. Let $\vec{z} = (z_1, \ldots, z_n)$ be any configuration. Assume that, without loss of generality, $\pi(k) < \pi(k+1)$. Then,

$$\begin{aligned}
&\big(\mathbf{F}_L[\pi(k)] \cdot \mathbf{F}_L[\pi(k+1)]\big)(\vec{z}) \\
&\quad = \mathbf{F}_L[\pi(k+1)]\big(\mathbf{F}_L[\pi(k)](\vec{z})\big) \\
&\quad = \mathbf{F}_L[\pi(k+1)](z_1, \ldots, f_{\pi(k)}(z_1, \ldots, z_n), \ldots, z_n) \\
&\quad = (z_1, \ldots, f_{\pi(k)}(z_1, \ldots, z_n), \ldots, f_{\pi(k+1)}(z_1, \ldots, f_{\pi(k)}(z_1, \ldots, z_n), \ldots, z_n), \ldots, z_n) \\
&\quad = \mathbf{F}_L[\pi(k)](z_1, \ldots, f_{\pi(k+1)}(z_1, \ldots, z_n), \ldots, z_n) \\
&\quad = \mathbf{F}_L[\pi(k)]\big(\mathbf{F}_[\pi(k+1)](\vec{z})\big) \\
&\quad = \big(\mathbf{F}_L[\pi(k+1)] \cdot \mathbf{F}_L[\pi(k)]\big)(\vec{z})
\end{aligned}$$

This proves the proposition. ■

5.2 The update graph

Given an undirected graph $G = (V, E)$, the *update graph* $U = U(G) = U(V, E)$ consists of vertex set S_V and edge set $\{(\pi, \pi') \mid \pi \text{ and } \pi' \text{ are adjacent}\}$.

Example: We determine the update graph for Circ_4 with $V = \{0, 1, 2, 3\}$.

Based on the update graph, we can define an equivalence relation on S_V with respect to $U = U(V, E)$:

$$\pi \sim_U \pi' \Longleftrightarrow_{\text{def}} \pi \text{ and } \pi' \text{ are connected by a path in } U$$

Proposition 5.3 *Let $G = (V, E)$ be an undirected graph, $\|V\| = n$. Let $\pi, \pi' \in S_V$ and let $U = U(V, E)$ be the update graph. If $\pi \sim_U \pi'$ then $\mathbf{F}_{(L,\pi)} = \mathbf{F}_{(L,\pi')}$ for all sets L of local transition functions compatible with G.*

Proof: The proof is by induction on the distance d between permutations in the update graph U. The distance $d_U(\pi, \pi')$ is defined to be the length of a shortest path from π to π' in U.

- *Base of induction*: Let $d = 0$. So, $d_U(\pi, \pi') = 0$, i.e., $\pi = \pi'$.
- *Induction step*: Let $d_U(\pi, \pi') = d_U(\pi', \pi) = d > 0$. Let $(\pi_0, \ldots, \pi_{d-1}, \pi_d)$ be a shortest path in U such that $\pi_0 = \pi'$ and $\pi_d = \pi$. It follows that π_{d-1} and π_d are adjacent with respect to U. Thus, there is a k such that $\{\pi(k), \pi(k+1)\} \notin E$ and $\pi(i) = \pi_{d-1}(i)$ for all $i \notin \{k, k+1\}$. We obtain for any set L of local transition functions and $\vec{z} \in D^n$

$$
\begin{aligned}
&\mathbf{F}_{(L,\pi)}(\vec{z}, n) \\
&= \left(\prod_{j=1}^{n} \mathbf{F}_L[\pi(j)] \right)(\vec{z}) \\
&= \left(\prod_{j=1}^{k-1} \mathbf{F}_L[\pi(j)] \cdot \mathbf{F}_L[\pi(k)] \cdot \mathbf{F}_L[\pi(k+1)] \cdot \prod_{j=k+2}^{n} \mathbf{F}_L[\pi(j)] \right)(\vec{z}) \\
&= \left(\prod_{j=1}^{k-1} \mathbf{F}_L[\pi(j)] \cdot \mathbf{F}_L[\pi(k+1)] \cdot \mathbf{F}_L[\pi(k)] \cdot \prod_{j=k+2}^{n} \mathbf{F}_L[\pi(j)] \right)(\vec{z}) \\
&\qquad\qquad \text{(by Proposition 5.2)} \\
&= \left(\prod_{j=1}^{k-1} \mathbf{F}_L[\pi_{d-1}(j)] \cdot \mathbf{F}_L[\pi_{d-1}(k)] \cdot \mathbf{F}_L[\pi_{d-1}(k+1)] \cdot \prod_{j=k+2}^{n} \mathbf{F}_L[\pi_{d-1}(j)] \right)(\vec{z}) \\
&= \left(\prod_{j=1}^{n} \mathbf{F}_L[\pi_{d-1}(j)] \right)(\vec{z}) \\
&= \mathbf{F}_{(L,\pi_{d-1})}(\vec{z}, n) \\
&= \mathbf{F}_{(L,\pi')}(\vec{z}, n) \qquad \text{(by induction assumption)}
\end{aligned}
$$

This proves the proposition. ∎

We consider equivalence classes $[\pi]_U$ of a permutation π with respect to $U = U(V, E)$, i.e.,

$$[\pi]_U =_{\text{def}} \{\pi' \mid \pi \sim_U \pi'\},$$

together with the quotient set with respect to the equivalence relation $\sim_U$

$$S_X / \sim_U = \{[\pi]_U \mid \pi \in S_X\}.$$

Proposition 5.4 *Let $G = (V, E)$ be an undirected graph and let $U = U(V, E)$ be the update graph. Then, there exists a bijective mapping*

$$f_G \;:\; S_V/\sim_U \;\;\to\;\; \mathrm{Acyc}(G),$$

where $\mathrm{Acyc}(G)$ *is the set of all acyclic orientations of* G.

Proof: We first construct an appropriate mapping $\tilde{f}_G : S_V \to \mathrm{Acyc}(G)$. Any permutation $\pi \in S_V$ induces a linear ordering $\leq_\pi$ on V by

$$i \leq_\pi j \iff_{\mathrm{def}} \pi(i) \leq \pi(j).$$

Any linear ordering $\leq_\pi$ on V induces an acyclic orientation: for each $\{i, j\} \in E$ set

$$i \to j \iff_{\mathrm{def}} i <_\pi j$$

Let $\tilde{f}_G$ map each permutation to the according orientation. We have to argue that $\tilde{f}_G(\pi) = \tilde{f}_G(\pi')$ for $\pi \sim_U \pi'$. It suffices to show $\tilde{f}_G(\pi) = \tilde{f}_G(\pi')$ for adjacent permutations π, π' (proof of the general case is then by induction): If π and π' are adjacent, they differ in exactly two consecutive entries not connected by an edge in E. Thus, $\tilde{f}_G(\pi) = \tilde{f}_G(\pi')$.

Now, define $f_G : S_V/\sim_U \to \mathrm{Acyc}(G)$ by $f_G([\pi]_U) =_{\mathrm{def}} \tilde{f}_G(\pi)$. Observe that f_G is injective (exercise!). It remains to show that f_G is surjective. Consider an acyclic orientation of G. For vertex $i \in V$ define the rank of i as follows:

$$\mathrm{rank}(i) \;=_{\mathrm{def}}\; \text{length of a longest directed path to } i \text{ (with respect to the given acyclic orientation)}$$

We should note that $\mathrm{rank}(i) = \mathrm{rank}(j)$ implies $\{i, j\} \notin E$ for $i \neq j$. We define

$$H =_{\mathrm{def}} \{h \mid \mathrm{rank}^{-1}(h) \neq \emptyset\}$$

and for $h \in H$

$$\mathrm{rnk}^{-1}(h) =_{\mathrm{def}} (i_1, \ldots, i_{m_h}),$$

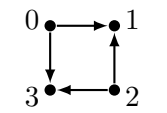
where $\mathrm{rank}(i_j) = h$ and $i_j < i_k$ for $j < k$. Furthermore, consider

$$\big[\big(\mathrm{rnk}^{-1}(0), \mathrm{rnk}^{-1}(1), \ldots, \mathrm{rnk}^{-1}(t)\big)\big]_U$$

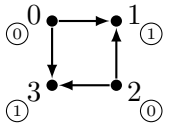
with $t = \max H$. Then, clearly, f_G maps $\big[\big(\mathrm{rnk}^{-1}(0), \ldots, \mathrm{rnk}^{-1}(t)\big)\big]_U$ to the given orientation. Thus, f_G is surjective. Hence, f_G is bijective. This proves the proposition. ∎

Example: Consider again Circ_4:

$$[(0,2,1,3)]_U \qquad \longmapsto$$



$$[(\underbrace{0,2}_{\mathrm{rnk}^{-1}(0)}, \underbrace{1,3}_{\mathrm{rnk}^{-1}(1)})]_U \qquad \longmapsto$$



Proposition 5.5 *For any undirected graph $G = (V, E)$, $\|V\| = n$, and any set L of local transition functions compatible with G,*

$$\| \left\{ \mathbf{F}_{(L,\pi)} \mid \pi \in S_V \right\} \| \leq \|\mathrm{Acyc}(G)\|;$$

and the bound is sharp.

Proof: Using Proposition 5.3 and Proposition 5.4, we obtain the following:

$$\left\| \left\{ \mathbf{F}_{(L,\pi)} \mid \pi \in S_V \right\} \right\| \leq \|\{[\pi]_U \mid \pi \in S_V\}\| = \|S_V / \sim_U\| = \|\mathrm{Acyc}(G)\|$$

Sharpness is left as an exercise. This proves the proposition. ■

> **Example:** It holds that $\|\mathrm{Acyc}(\mathrm{Circ}_n)\| = 2^n - 2$, since only two of the 2^n possible orientations of Circ_n are not acyclic. Thus, there are at most $2^n - 2$ essentially different global network maps.

5.3 Acyclic orientations and the chromatic polynomial

How to compute $\|\mathrm{Acyc}(G)\|$?

Let $G = (V, E)$ be an undirected graph. A *vertex coloring* with k colors $1, \ldots, k$ is a mapping $f : V \to \{1, \ldots, k\}$ such that $f(u) \neq f(v)$ if $\{u, v\} \in E$. Define $P_G(k)$ to be the number of different vertex colorings with k colors of G. The pausible choices for the number of colors are $0, 1, \ldots, n$. Thus, we know the function values of P_G for $n + 1$ arguments. Hence, there is a uniquely determined normal polynomial (i.e., the leading coefficient in the expanded form of the polynomial is 1) of degree n which takes on these specified function values. We identify P_G with this polynomial, and we call $P_G(x)$ the *chromatic polynomial* of graph G.

> **Example:** Let $G = K^n$. It holds that $P_G(k) = 0$ for $k \in \{0, 1, \ldots, n-1\}$. Moreover, $P_G(n) = n!$. Thus, the chromatic polynomial of G is given by
>
> $$P_G(x) = \prod_{j=0}^{n-1} (x - j).$$

Lemma 5.6 *Let G, H be undirected graphs.*

1. *If G is a one-vertex graph, $P_G(x) = x$.*
2. *$P_{G \oplus H}(x) = P_G(x) \cdot P_H(x)$*
3. *$P_G(x) = P_{G-e}(k) - P_{G/e}(x)$*

Example: Let T be a tree with n vertices. Let u be an arbitrary leaf of T and $e = \{u, v\}$ be the edge connecting u with T. Then, it holds

$$\begin{aligned} P_T(x) &= P_{T-e}(x) - P_{T/e}(x) \\ &= P_{T'}(x) \cdot x - P_{T''(x)} \end{aligned}$$

Here, T' is a tree with $n-1$ vertices, T'' is a tree with $n-1$ vertices. Actually, $T' \simeq T''$. We conclude

$$P_T(x) = P_{T'}(x) \cdot (x-1).$$

By iteration, we obtain $P_T(x) = x(x-1)^{n-1}$.

Thus, each tree with n vertices has the same chromatic polynomial independent of its structure. Moreover, a graph G with n vertices is a tree if and only if $P_G(x) = x(x-1)^{n-1}$.

Lemma 5.7 *Let G be an undirected graph. Suppose there are graphs G_1, G_2 such that $G = G_1 \cup G_2$ and $G_1 \cap G_2 = K^n$. Then,*

$$P_G(x) = \frac{P_{G_1}(x) \cdot P_{G_2}(x)}{P_{K^n}(x)}$$

Proof: Each vertex coloring f of G corresponds to exactly one pair (f_1, f_2) of colorings of G_1 and G_2 which are identical on K^n. So, let f_1 be a k-coloring of G_1. Then, there are $P_{G_2}(k)/P_{K^n}(k)$ k-colorings of G_2 which are identical on K^n with f_1. This proves the lemma. ■

Example: We want to compute, once more, the chromatic polynomial for K^n. We start with the following recursion:

$$\begin{aligned} P_{K^n}(x) &= P_{K^n-e}(x) - P_{K^n/e}(x) \\ &= \frac{P_{K^{n-1}}(x)^2}{P_{K^{n-2}}(x)} - P_{K^{n-1}}(x) \\ &= \frac{P_{K^{n-1}}(x)}{P_{K^{n-2}}(x)} \; (P_{K^{n-1}}(x) - P_{K^{n-2}}(x)) \end{aligned}$$

By induction we can prove that $P_{k^n}(x) = x^{\underline{n}}$:

- *Base of induction*: We have two case here, $n \in \{1, 2\}$: $P_{K^1}(x) = x = x^{\underline{1}}$ and $P_{K^2}(x) = x(x-1) = x^{\underline{2}}$.
- *Induction step*: For $n > 2$, we have

$$\begin{aligned} P_{K^n}(x) &= \frac{x^{\underline{n-1}}}{x^{\underline{n-2}}} \cdot \left(x^{\underline{n-1}} - x^{\underline{n-2}}\right) && \text{(by induction assumption)} \\ &= (x - (n-1) + 1) \cdot x^{\underline{n-2}} \cdot ((x - (n-1) + 1) - 1) \\ &= x^{\underline{n-2}} \cdot (x - (n-2)) \cdot (x - (n-1)) \\ &= x^{\underline{n}} \end{aligned}$$

We give a different interpretation of $P_G(x)$.

Proposition 5.8 *Let $G = (V, E)$ be an undirected graph. Then, $P_G(k)$ is equal to the numbers of pairs (f, O) where $f : V \to \{1, \ldots, k\}$ and O is an orientation of G such that*

1. *the orientation O is acyclic,*
2. *if $u \to v$ in orientation O then $f(u) > f(v)$.*

Proof: Consider a pair (f, O) satisfying (i), (ii). From (ii) it follows that $f(u) \neq f(v)$ for $\{u, v\} \in E$. Thus, f is a vertex coloring with k colors. Moreover, (ii) implies (i). Conversely, if f is a vertex coloring with k colors then f defines a unique acyclic orientation O by $u \to v$ if and only if $f(u) > f(v)$. Hence, the number of allowed pairs (f, O) is the number of vertex colorings with colors $1, \ldots, k$ and is, thus, $P_G(k)$. ■

Proposition 5.8 suggests the following modification: Let $G = (V, E)$ be an undirected graph and let $k \in \{1, \ldots, n\}$ where $n = \|V\|$. Define $\overline{P}_G(k)$ to be the number of pairs (f, O) where $f : V \to \{1, \ldots, k\}$ and O is an orientation of G such that:

1. the orientation O is acyclic,
2. if $u \to v$ in orientation O then $f(u) > f(v)$.

We say that the function f is compatible with O if f satisfies the second conditions.

Lemma 5.9 *Let G, H be undirected graphs.*

1. *If G is one-vertex graph then $\overline{P}_G(x) = x$.*
2. *$\overline{P}_{G \oplus H}(x) = \overline{P}_G(x) \cdot \overline{P}_H(x)$*
3. *$\overline{P}_G(x) = \overline{P}_{G-e}(x) + \overline{P}_{G/e}(x)$ for any $e \in E$*

Proof: The first two statements are obvious.

In order to show the third statement, let $f : V \to \{1, \ldots, k\}$ be a mapping and let O be an acyclic orientation of $G-e$ compatible with f, where $e = \{u, v\} \in E$. Let O_1 be the orientation of G obtained by adjoining $u \to v$ to O, and O_2 that is obtained by adjoining $v \to u$ to O. We show that for each pair (f, O) exactly one of O_1 and O_2 is an acyclic orientation compatible with f, except for $\overline{P}_{G/e}(k)$ of the pairs, in which case both O_1 and O_2 are acyclic orientations compatible with f. Thus, $\overline{P}_{G-e}(k) = \overline{P}_G(k) - \overline{P}_{G/e}(k)$. We consider the following three cases:

- If $f(u) > f(v)$ then O_2 is not compatible with f while O_1 is compatible. Moreover, O_1 is acyclic, since if $u \to v \to w_1 \to w_2 \to \cdots \to u$ were a directed cycle in O_1, we would have $f(u) > f(v) \geq f(w_1) \geq f(w_2) \geq \cdots \geq f(u)$, which is a contradiction.

- If $f(u) < f(v)$ then we can argue symmetrically to the first case.
- If $f(u) = f(v)$, both O_1 and O_2 are compatible with f. Then, at least one of them is acyclic; if not: O_1 contains a cycle $u \to v \to w_1 \to w_2 \to \cdots \to u$ and O_2 contains a cycle $v \to u \to w_1' \to w_2' \to \cdots \to v$. Hence, O contains a cycle $v \to w_1 \to w_2 \to \cdots \to u \to w_1' \to w_2' \to \cdots \to u$ which is not possible.

It remains to prove that O_1 and O_2 are acyclic for exactly $\overline{P}_{G/e}(k)$ pairs (f, O) with $f(u) = f(v)$. Define $\Phi(f, O) =_{\text{def}} (f', O')$ such that $f' : V(G/e) \to \{1, \ldots, k\}$ (note that $f(u) = f(v)$) and O' is an acyclic orientation of G/e compatible with f'. Let z be the vertex obtained by identifying u and v. Define f' to be the following function:

$$f'(w) =_{\text{def}} \begin{cases} f(w) & \text{if } w \in V \setminus \{u, v\} \\ f(u) & \text{if } w = z \end{cases}$$

Define O' by $w_1 \to w_2$ in O' if and only if $w_1 \to w_2$ in O. Then, Φ is a bijection. This proves the proposition. ■

Theorem 5.10 (Stanley 1973) *For each graph $G = (V, E)$ such that $\|V\| = n$,*

$$\overline{P}_G(x) = (-1)^n P_G(-x).$$

Proof: Using the recursive rules according to Lemma 5.9 and Lemma **??**, we prove the statement by induction on the number n of vertices.

- *Base of induction*: Let $n = 1$. Then, $\overline{P}_G(x) = x = (-1)^1(-x) = (-1)^1 P_G(-x)$.
- *Induction step*: Suppose $n > 1$. Again, we argue inductively, in this case however, on the number of edges. For the base of induction, let G be the empty graph on n vertices. Then, $\overline{P}_G(x) = x^n = (-1)^n(-x)^n = (-1)^n P_G(-x)$. For the induction step, suppose $\|E\| \geq 1$. Then, for some edge $e \in E$

$$\begin{aligned} \overline{P}_G(x) &= \overline{P}_{G-e}(x) + \overline{P}_{G/e}(x) \\ &= (-1)^n \; P_{G-e}(-x) + (-1)^{n-1} \; P_{G/e}(-x) \\ &= (-1)^n \; \big(P_{G-e}(-x) - P_{G/e}(-x)\big) \\ &= (-1)^n \; P_G(-x) \end{aligned}$$

This proves the theorem. ■

Corollary 5.11 $\|\text{Acyc}(G)\| = (-1)^n \; P_G(-1)$.

Proof: It holds that $\|\mathrm{Acyc}(G)\| = \overline{P}_G(1) = (-1)^n \; P_G(-1)$. ■

Example: We want to compute $\|\mathrm{Acyc}(\mathrm{Circ}_n)\|$ for $n \geq 3$. First, we prove that $P_{\mathrm{Circ}_n}(x) = (x-1)^n + (-1)^n(x-1)$ by induction on $n \geq 3$.

- *Base of induction*: For $n = 3$, we calculate

$$\begin{aligned} P_{\mathrm{Circ}_3}(x) &= x(x-1)(x-2) \\ &= x^3 - 3x^2 + 2x \\ &= x^3 - 3x^2 + 3x - 1 - (x-1) \\ &= (x-1)^3 + (-1)^3(x-1) \end{aligned}$$

- *Induction step*: For $n > 3$, we calculate

$$\begin{aligned} P_{\mathrm{Circ}_n}(x) &= P_{\mathrm{Circ}_n - e}(x) - P_{\mathrm{Circ}_n / e}(x) \\ &= x(x-1)^{n-1} - \big((x-1)^{n-1} + (-1)^{n-1}(x-1)\big) \\ &= (x-1)^n(x-1) - (-1)^{n-1}(x-1) \\ &= (x-1)^n + (-1)^n(x-1) \end{aligned}$$

Now, from Corollary 5.11, we obtain $\|\mathrm{Acyc}(\mathrm{Circ}_n)\| = 2^n - 2$ by considering two distinctive cases:

- If n is even then $\overline{P}_{\mathrm{Circ}_n}(1) = P_{\mathrm{Circ}_n}(-1) = 2^n - 2$
- If n is odd then $\overline{P}_{\mathrm{Circ}_n}(1) = -P_{\mathrm{Circ}_n}(-1) = -\left(-2^n - (-2)\right) = 2^n - 2$

Note that, unless P = NP, there is no algorithm for computing the number of acyclic orientations of a given graph with n vertices, which runs in time polynomial in n.

Potential models 6

6.1 Network games

Models based on maximizing a certain potential function are typical for rational actors (aka agents). They belong to a class of mechanisms where actors choose their decisions on changing attribute values depending on the decisions of other actors in such a way they maximize their benefits, utilities, or preferences. In the following, we use game theory to analyze such models.

6.1.1 Games with utilities

Definition 6.1 *A* game with utilities Γ *is a triple* $(A, (S_1, \ldots, S_m), (u_1, \ldots, u_m))$*, where*

1. $A = \{1, \ldots, m\}$ *is a finite, non-empty set of* agents*,*
2. S_i *is a non-empty set of* strategies *of agent* $i \in A$*, and*
3. $u_i : S_1 \times \cdots \times S_m \to \mathbb{R}$ *is a utility function for agent* i*.*

According to the definiton above, we introduce some notations:

- $S =_{\text{def}} \bigtimes_{k=1}^{m} S_k$ denotes the set of all strategy profiles of all agents; $S_{-i} =_{\text{def}} \bigtimes_{\substack{k=1 \\ k \neq i}}^{m} S_k$ denotes the set of all strategy profiles of all agents except agent i.
- For a strategy profile $s = (s_1, \ldots, s_m) \in S$, let s_{-i} denote the $(m-1)$-tuple consisting of strategies of all agents except agent i, i.e., $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_m)$.
- So, $s = (s_i, s_{-i})$ and $S = S_i \times S_{-i}$, by convention.
- We use $u = (u_1, \ldots, u_m) : S \to \mathbb{R}^m$ to denote the vector utility function, and we use $u_i(s) = u_i(s_1, \ldots, s_m) = u_i(s_i, s_{-i})$ to denote agent's i utility of a strategy profile

We consider a game Γ as a *one-shot non-cooperative game*. Each agent u chooses a strategy $s_i \in S_i$ independently of other agents and without knowing the choices of the other agents. The result is a strategy profile $s = (s_1, \ldots, s_m)$. Each agent i evaluates strategy profile s according to the utility function u_i.

A notion central to game theory is the Nash equilibrium.

Definition 6.2 *Let $\Gamma = (A, S, u)$ be a game with utilities, involving m agents. A strategy profile $s^* = (s_1^*, \ldots, s_m^*)$ is called* Nash equilibrium *if and only if $u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$ for all $s_i \in S_i$ and all $i \in A$.*

Intuitively, in a Nash equilibrium, no agent has an incentive to deviate from the chosen strategy.

Example: We exemplify the notions for three standard games.

- *Battle of sexes*: Male M and Female F want to spend time together, i.e., $A = \{M, F\}$. Alternatives are cinema (c) or football (f). So, the sets of strategies for both are $S_M = S_F = \{c, f\}$. The set of strategy profiles is

$$S = S_F \times S_M = \{\ (c,c), (c,f), (f,c), (f,f)\ \}$$

  where the first component of a pair denotes Female's strategy and the second component is Male's strategy. Now, on the one hand-side, Male prefers football over cinema but together is better than alone. So, M's preference can be described by the following utility function:

$$u_M : \begin{array}{rcl} (f,f) & \mapsto & 3 \\ (c,c) & \mapsto & 2 \\ (c,f) & \mapsto & 1 \\ (f,c) & \mapsto & 0 \end{array}$$

  On the other hand-side, Female prefers cinema over football but together is better than alone. So, F's utilities could be as follows:

$$u_F : \begin{array}{rcl} (c,c) & \mapsto & 3 \\ (f,f) & \mapsto & 2 \\ (c,f) & \mapsto & 1 \\ (f,c) & \mapsto & 0 \end{array}$$

  Combined, both utility functions can be modelled as a payoff (bi-)matrix:

$$\begin{array}{cc} & \begin{array}{cc} M & \\ f & c \end{array} \\ \begin{array}{c} F \\ \begin{array}{c} f \\ c \end{array} \end{array} & \left(\begin{array}{cc} (2,3) & (0,0) \\ (1,1) & (3,2) \end{array} \right) \end{array}$$

  Since all information on the game is contained in this representation, we will also identify such a matrix with a 2-person game.

Which strategy profiles are Nash equilibria? We examine all strategy profiles individually:

- (c, c) is a Nash equilibrium, since

$$u_F(c, c) = 3 > 0 = u_F(f, c)$$
$$u_M(c, c) = 2 > 1 = u_M(c, f)$$

- (c, f) is not a Nash equilibrium, since

$$u_F(c, f) = 1 < 2 = u_F(f, f)$$

- (f, c) is not a Nash equilibrium, since

$$u_M(f, c) = 0 < 3 = u_M(f, f)$$

- (f, f) is a Nash equilibrium, since

$$u_F(f, f) = 2 > 1 = u_F(c, f)$$
$$u_M(f, f) = 3 > 0 = u_M(f, c)$$

Now, suppose Female is more decisive: she excludes football an option. Thus, F's modified utility function leads to the following (bimatrix) game

$$\begin{pmatrix} (1,3) & (0,0) \\ (2,1) & (3,2) \end{pmatrix}$$

Then, the only Nash equilibrium is (c, c).

- *Prisoner's dilemma*: Bonnie and Clyde have been captivated and charged with bank robbery. However, the prosecutor is only able to prove illegal possession of firearms to them; without confessions, the sentence will then be 3 years in prison. If one of them makes a confession then the confessor will be sentenced to one year and the non-confessor will be sentenced to 9 years in prison. If both confess then they will be sentenced to 7 years in prison, respectively.

  A game-based formulation of this decision scenario is given by the following game with utilities:

$$\begin{array}{c} \\ s_{11} \\ s_{12} \end{array} \begin{array}{c} \begin{array}{cc} s_{21} & s_{22} \end{array} \\ \begin{pmatrix} (2,3) & (0,0) \\ (1,1) & (3,2) \end{pmatrix} \end{array}$$

  where s_{i1} stands for strategy "confession" and s_{i2} stands for "no confession."

  Which strategy profiles are Nash equilibria?

  - (s_{11}, s_{21}) is a Nash equilibrium, since

$$u_1(s_{11}, s_{21}) = -7 > -9 = u_1(s_{12}, s_{21})$$
$$u_2(s_{11}, s_{21}) = -7 > -9 = u_2(s_{11}, s_{22})$$

– (s_{11}, s_{22}) is not a Nash equilibrium, since

$$u_2(s_{11}, s_{22}) = -9 < -7 = u_2(s_{11}, s_{21})$$

– (s_{12}, s_{21}) is not a Nash equilibrium, since

$$u_1(s_{12}, s_{21}) = -9 < -7 = u_1(s_{11}, s_{21})$$

– (s_{12}, s_{22}) is not a Nash equilibrium, since

$$u_1(s_{12}, s_{22}) = -3 < -1 = u_1(s_{11}, s_{22})$$

Why is this game a dilemma? Because (s_{12}, s_{21}) would be a better strategy profile for both. But it is no equilibrium; each agent could be better off when changing the strategy. The reason for that is the lack of communication and coordination.

- *Rock-paper-scissor*: The scenario consists of two players each of them chooses one of the three gestures "rock", "paper", or "scissor" as a strategy. The rules of winning the game are as follows:
  – rock defeats scissor
  – scissor defeats paper
  – paper defeats rock

  The loser of a game pays a unit to the winner. We can express this a game with utilities by the following bimatrix game:

$$\begin{array}{r} \\ \text{rock} \\ \text{paper} \\ \text{scissor} \end{array} \begin{array}{c} \begin{array}{ccc} \text{rock} & \text{paper} & \text{scissor} \end{array} \\ \begin{pmatrix} (0,0) & (-1,1) & (1,-1) \\ (1,-1) & (0,0) & (-1,1) \\ (-1,1) & (1,-1) & (0,0) \end{pmatrix} \end{array}$$

  Obviously, there is no Nash equilibrium for this game in pure strategies.

An alternative characterization of Nash equilibria can be given by best-response dynamics.

Definition 6.3 *Let $\Gamma = (A, S, u)$ be a game with utilities.*

1. *The* best response *(map) $\beta_i : S_{-i} \to \mathcal{P}(S_i)$* for agent *$i \in A$ is defined by*

$$\beta_i(s_{-i}) =_{\text{def}} \left\{ s_i \in S_i \;\middle|\; u_i(s_i, s_{-i}) = \max_{s_i' \in S_i} u_i(s_i', s_{-i}) \right\}$$

2. *The* best response *$\beta : S \to \bigtimes_{i=1}^{m} \mathcal{P}(S_i)$ is defined by*

$$\beta(s) =_{\text{def}} \beta_1(s_{-1}) \times \cdots \times \beta_n(s_{-n}).$$

Theorem 6.4 *Let $\Gamma = (A, S, u)$ be a game with utilities. For all $s^* \in S$, it holds*

$$s^* \text{ is a Nash equilibrium} \iff s^* \in \beta(s^*).$$

Proof: Let $s^* \in S$ be a strategy profile. Then, the following chain of equivalences holds:

$$\begin{aligned}
&s^* \text{ is a Nash equilibrium} \\
&\iff u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*) \text{ for all } i \in A,\ s_i \in S_i && \text{(by Definition 6.2)} \\
&\iff u_i(s_i^*, s_{-i}^*) = \max_{s_i \in S_i} u_i(s_i, s_{-i}^*) \text{ for all } i \in A,\ s_i \in S_i \\
&\iff s_i^* \in \beta_i(s_{-i}^*) \text{ for all } i \in A && \text{(by Definition 6.3.1)} \\
&\iff s^* \in \beta(s^*) && \text{(by Definition 6.3.2)}
\end{aligned}$$

This proves the theorem. ■

Example: Consider the following payoff matrix for a two-person game with identical utility function

$$\begin{array}{c} \\ s_{11} \\ s_{12} \end{array} \begin{array}{c} \begin{array}{cc} s_{21} & s_{22} \end{array} \\ \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix} \end{array}$$

The best responses for the agents are

$$\begin{aligned}
\beta_1(s_{21}) &= \{s_{11}, s_{12}\}, & \beta_1(s_{22}) &= \{s_{11}\} \\
\beta_2(s_{11}) &= \{s_{22}\}, & \beta_2(s_{12}) &= \{s_{22}\}
\end{aligned}$$

So, the best response is:

$$\begin{aligned}
\beta(s_{11}, s_{21}) &= \{s_{11}, s_{12}\} \times \{s_{22}\} \\
\beta(s_{11}, s_{22}) &= \{s_{11}\} \times \{s_{22}\} \\
\beta(s_{12}, s_{21}) &= \{s_{11}, s_{12}\} \times \{s_{22}\} \\
\beta(s_{12}, s_{22}) &= \{s_{11}\} \times \{s_{22}\}
\end{aligned}$$

By Theorem 6.4, (s_{11}, s_{22}) is a unique Nash equilibrium.

6.1.2 Potential games

An important class of games with equilibrium guarantee is the class of potential games.

Definition 6.5 *Let $\Gamma = (A, S, u)$ be a game with utilities, and let $P : S \to \mathbb{R}$ be any function.*

1. *P is said to be an* ordinal potential function *for Γ if and only if for all $i \in A$, $s_{-i} \in S_{-i}$, $s_i, \bar{s}_i \in S_i$,*

$$u_i(s_i, s_{-i}) - u_i(\bar{s}_i, s_{-i}) > 0 \iff P(s_i, s_{-i}) - P(\bar{s}_i, s_{-i}) > 0.$$

   Γ is said to be an ordinal potential game *if and only if there is an ordinal potential function for Γ.*

2. *P is said to be a* potential function *for Γ if and only if for all $i \in A$, $s_{-i} \in S_{-i}$, $s_i, \bar{s}_i \in S_i$,*

$$u_i(s_i, s_{-i}) - u_i(\bar{s}_i, s_{-i}) = P(s_i, s_{-i}) - P(\bar{s}_i, s_{-i}).$$

   Γ is said to be a potential game *if and only if there is a potential function for Γ.*

Example: We discuss the notions for two games.

- Consider the following bimatrix game:

$$\Gamma = \begin{pmatrix} (0,3) & (1,2) \\ (3,1) & (2,0) \end{pmatrix}$$

  According to Definition 6.5, it suffices to consider the following differences:

$$\begin{aligned} u_1(s_{11}, s_{21}) - u_1(s_{12}, s_{21}) &= -3 \\ u_1(s_{11}, s_{22}) - u_1(s_{12}, s_{22}) &= -1 \\ u_2(s_{11}, s_{21}) - u_2(s_{11}, s_{22}) &= 1 \\ u_2(s_{12}, s_{21}) - u_2(s_{12}, s_{22}) &= 1 \end{aligned}$$

  Then, Γ is an ordinal potential game. An ordinal potential function P is represented by the matrix

$$\Gamma = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix},$$

  since

$$\begin{aligned} P(s_{11}, s_{21}) - P(s_{12}, s_{21}) &= -1 < 0 \\ P(s_{11}, s_{22}) - P(s_{12}, s_{22}) &= -1 < 0 \\ P(s_{11}, s_{21}) - P(s_{11}, s_{22}) &= 1 > 0 \\ P(s_{12}, s_{21}) - P(s_{12}, s_{22}) &= 1 > 0 \end{aligned}$$

  However, Γ is not a potential game. (An explanation will be given later.)

- Recall that the *prisoner's dilemma* can be represent by the following bimatrix game:

$$\Gamma = \begin{pmatrix} (-7,-7) & (-1,-9) \\ (-9,-1) & (-3,-3) \end{pmatrix}$$

Γ is a potential game, where the potential function P is given by

$$P = \begin{pmatrix} 4 & 2 \\ 2 & 0 \end{pmatrix}.$$

Proposition 6.6 *Let $\Gamma = (A, S, u)$ be a game with utilities and an ordinal potential function P, and let $s^* \in S$. Then, s^* is a Nash equilibrium if and only if for all $i \in A$ and $s_i \in S_i$, it holds that*

$$P(s^*) \geq P(s_i, s^*_{-i}).$$

Proof: Immediate from Definition 6.5. ■

Corollary 6.7 *Each finite ordinal potential game has a Nash equilibrium.*

Proof: For a finite ordinal potential game Γ, it's ordinal potential funciton P has a maximum. Let $s^* \in S$ be such that $P(s^*)$ is maximum. Then, s^* is a Nash equilibrium by Proposition 6.6. ■

Though we have a certain flexibility in choosing an ordinal potential for an ordinal potential game, in case of exact potentials it is less so: they are unique up to some additive constant.

Proposition 6.8 *Let $\Gamma = (A, S, u)$ be a potential game with potentials P_1 and P_2. Then, there is a $c \in \mathbb{R}$ such that for all $s \in S$,*

$$P_1(s) - P_2(s) = c$$

Proof: Choose any $s^* \in S$. Define for all $s \in S$,

$$H(s) =_{\text{def}} \sum_{i=1}^{m} \left(u_i(t^{i-1} - u_i(t^i)\right),$$

where $t^0 = s$ and $t^i = (s^*_i, t^{i-1}_{-i})$ for $i \in \{1, \ldots, m\}$. For each potential P for Γ we have

$$H(s) = \sum_{i=1}^{m} \left(u_i(t^{i-1}) - u_i(t^i)\right) = \sum_{i=1}^{m} \left(P(t^{i-1}) - P(t^i)\right) = P(t^0) - P(t^m) = P(s) - P(s^*).$$

Hence,

$$P_1(s) - P_2(s) = H(s) + P_1(s^*) - (H(s) + P_2(s^*)) = P_1(s^*) - P_2(s^*).$$

The last difference is constant. This proves the proposition. ■

How can we decide whether a given game with utilites is, in fact, a potential game? To answer this question, we give a characterization based on the structure of utility functions. It is helpful to introduce some additional notions.

Let $\Gamma = (A, S, u)$ be a game with utilites.

A sequence $p = (s^0, s^1, \ldots, s^N)$ is a *path* in Γ if and only if for all $k \geq 1$, there is an $i \in A$ such that $s^k = (s_i, s_{-i}^{k-1})$ for some $s_i \in S_i$ with $s_i \neq s_i^{k-1}$. The agent $i \in A$ is then called the *deviator* for k. A path $p = (s^0, s^1, \ldots, s^N)$ is said to be *closed* iff $s^0 = s^N$. A path $p = (s^0, s^1, \ldots, s^N)$ is said to be *simple* iff $s^j \neq s^k$ for all $0 \leq j < k \leq N - 1$.

Furthermore, for a finite path $p = (s^0, s^, \ldots, s^N)$ in Γ, define

$$I(\Gamma, p) =_{\text{def}} \sum_{k=1}^{N} \left(u_{i_k}(s^k) - u_{i_k}(s^{k-1}) \right),$$

where i_k is the deviator for k.

Theorem 6.9 *Let $\Gamma = (A, S, u)$ be a game with utilities. The following statements are equivalent:*

1. *Γ is a potential game.*
2. *$I(\Gamma, p) = 0$ for each finite, closed path p in Γ.*
3. *$I(\Gamma, p) = 0$ for each finite, simple, closed path p in Γ.*
4. *$I(\Gamma, p) = 0$ for each finite, simple, closed path p in Γ of length 4.*

Proof: We show the following implications:

- (1) ⇒ (2): Let P be a potential function for $\Gamma = (A, S, u)$. Let $p = (s^0, s^1, \ldots, s^N)$ be a closed path. Then, we conclude

$$\begin{aligned} I(\Gamma, p) &= \sum_{k=1}^{N} \left(u_{i_k}(s^k) - u_{i_k}(s^{k-1}) \right) \\ &= \sum_{k=1}^{N} \left(P(s^k) - P(s^{k-1}) \right) \qquad \text{(since } P \text{ is a potential function for } \Gamma) \\ &= P(s^N) - P(s^0) \\ &= 0 \qquad \text{(since } s^N = s^0) \end{aligned}$$

- (2) ⇒ (1): Fix an arbitrary strategy profile $z \in S$. For $s \in S$, let $p(s) = (s^0, \dots, s^N)$ denote an arbitrary path from $s^0 = z$ to $s^N = s$. We define

$$P(s) =_{\text{def}} I(\Gamma, p(s)).$$

  Note that there is always a path from z to a strategy profile s. We have to show that the following two statements are true:

  1. P is well-defined, i.e., the definition of P is independent of the choice of the path $p(s)$.
  2. P is a potential function for Γ

  This can be seen as follows:

  1. Let $q(s) = (s^0, \dots t^M)$ be another path such that $t^0 = z$ and $t^M = s$. Then, the concatenated path $\gamma = (s^0, \dots, s^N, t^{M-1}, \dots, t^0)$ is a closed path in Γ. By our assumption, it holds that $I(\Gamma, \gamma) = 0$. We conclude

$$\begin{aligned} I(\Gamma, p(s)) &= -I\left(\Gamma, (s^N, t^{M-1}, \dots, t^0)\right) \\ &= I\left(\Gamma, (t^0, \dots, t^{M-1}, s^N)\right) \\ &= I(\Gamma, q(s)) \end{aligned}$$

  2. For $i \in A$, let $s_i, s_i' \in S_i$ be two strategies, let $s_{-i} \in S_{-i}$. Again by our assumption, we obtain

$$\begin{aligned} 0 &= I\left(\Gamma, \left((s_i, s_{-i}), \dots, z, \dots, (s_i', s_{-i}), (s_i, s_{-i})\right)\right) \\ &= I\left(\Gamma, ((s_i, s_{-i}), \dots, z)\right) + I\left(\Gamma, \left(z, \dots, (s_i', s_{-i}), (s_i, s_{-i})\right)\right) \\ &= -I\left(\Gamma, (z, \dots, (s_i, s_{-i}))\right) + I\left(\Gamma, \left(z, \dots, (s_i', s_{-i})\right)\right) + \\ &\qquad + u_i(s_i, s_{-i}) - u_i(s_i', s_{-i}) \end{aligned}$$

  Consequently,

$$\begin{aligned} u_i(s_i, s_{-i}) - u_i(s_i', s_{-i}) &= I(\Gamma, (z, \dots, (s_i, s_{-i}))) - I(\Gamma, (z, \dots, (s_i', s_{-i}))) \\ &= P(s_i, s_{-i}) - P(s_i', s_{-i}) \end{aligned}$$

  Hence, P is a potential function.

- (2) ⇒ (3): Trivial.

- (3) ⇒ (4): Trivial.

- (4) ⇒ (2): Suppose $I(\Gamma, p) = 0$ for all simple, closed paths p of length 4 in $\Gamma = (A, S, u)$. We show that $I(\Gamma, p) = 0$ for all closed paths p of length N in Γ by induction on N:

  – *base of induction* $N \leq 4$: Cases $N \in \{1, 2, 3\}$ are trivial (in particular, there are no closed paths of odd lengths); for $N = 4$, the statement holds by the assumption.

– *inductive step* $N > 4$: Let $p = (s^0, s^1, \ldots, s^N)$ be a closed path with $N \geq 5$. Let $(i_1, \ldots, i_N)$ be the sequence of deviators for each step, i.e., $s^j = (s_{i_j}, s^{j-1}_{-i_j})$ such that $s_{i_j} \neq s^{j-1}_{i_j}$. Without loss of generality, assume $i_1 = 1$. Since $s^N = s^0$, there is $2 \leq j \leq N$ such that $i_j = 1$ and $s^j_{i_j} = s^0_1$. Choose j to be minimal subject to this condition, i.e., there is no $2 \leq k < j$ satisfying $i_k = 1$ and $s^k_{i_k} = s^0_1$.
First, suppose $j = 2$. That is, $s^2 = s^0$. Consider the path $q =_{\text{def}} (s^2, \ldots, s^N)$ of length $N - 1$. Then,

$$\begin{aligned} I(\Gamma, p) &= I(\Gamma, q) + u_1(s^2) - u_1(s^1) + u_1(s^1) - u_1(s^0) \\ &= u_1(s^2) - u_1(s^0) && \text{(by inductive assumption)} \\ &= 0 && \text{(since } s^2 = s^0\text{)} \end{aligned}$$

Now, suppose $j \geq 3$, i.e, $j \in \{3, \ldots, N\}$. Then, we have two subcases:

1. Subcase $i_{j-1} = i_j$. Consider path $q =_{\text{def}} (s^0, \ldots, s^{j-2}, s^j, \ldots, s^N)$. It holds

$$\begin{aligned} & I(\Gamma, q) \\ &= I\left(\Gamma, (s^0, \ldots, s^{j-2})\right) + u_{i_j}(s^j) - u_{i_j}(s^{j-2}) + I\left(\Gamma, (s^j, \ldots, s^N)\right) \\ &= I\left(\Gamma, (s^0, \ldots, s^{j-2})\right) + u_{i_j}(s^j) - u_{i_j}(s^{j-1}) + \\ &\qquad + u_{i_j}(s^{j-1}) - u_{i_j}(s^{j-2}) + I\left(\Gamma, (s^j, \ldots, s^N)\right) \\ &= I\left(\Gamma, (s^0, \ldots, s^{j-2})\right) + u_{i_j}(s^j) - u_{i_j}(s^{j-1}) + \\ &\qquad + u_{i_{j-1}}(s^{j-1}) - u_{i_{j-1}}(s^{j-2}) + I\left(\Gamma, (s^j, \ldots, s^N)\right) \\ &\qquad\qquad \text{(since } i_{j-1} = i_j\text{)} \\ &= I\left(\Gamma, (s^0, \ldots, s^{j-2}, s^{j-1}, s^j, \ldots, s^N)\right) \\ &= I(\Gamma, p) \end{aligned}$$

   Thus, by the inductive assumption, $I(\Gamma, p) = I(\Gamma, q) = 0$.

2. Subcase $i_{j-1} \neq i_j$. That is, we have the following scenario: ... Define a path $q_j = \text{def}(s^0, \ldots, s^{j-2}, t^{j-1}, s^j, \ldots, s^N)$ where $t^{j-1} = (s_{i_j}, s^{j-2}_{-i_j})$, i.e., the deviator in $(j-1)$-st step is 1. Now, path $r =_{\text{def}} (s^{j-2}, t^{j-1}, s^j, s^{j-1}, s^{j-2})$ is simple (since $i_j \neq i_{j-1}$), closed, and has length 4. Hence, $I(\Gamma, r) = 0$. That is,

$$I\left(\Gamma, (s^{j-2}, s^{j-1}, s^j)\right) = I\left(\Gamma, (s^{j-2}, t^{j-1}, s^j)\right).$$

   Therefore, $I(\Gamma, p) = I(\Gamma, q_j)$.
   Recursively repeated, we obtain a sequence of paths $q_j, q_{j-1}, \ldots, q_3$ such that $I(\Gamma, p) = I(\Gamma, q_k)$ for all $k \in \{3, \ldots, j\}$ and the deviator in q_k's step $k - 1$ is 1. The path q_3 corresponds to the case $j = 2$ above. Thus,

$$I(\Gamma, p) = I(\Gamma, q_3) = 0.$$

This proves the theorem. ■

We want to characterize potential games from a dynamical perspective.

Let $\Gamma = (A, S, u)$ be a game with utilities. Let $(s^t)_{t\in I}$ be any finite or infinite sequence of strategy profiles, i.e., $I = \mathbb{N}$ or $I = \{0, 1, \dots, n\}$ for some $n \in \mathbb{N}$. Then, the sequence $(s^t)_{t\in I}$ is called an *improvement path* if and only if for all $t \in I, t > 0$, there is an $i \in A$ such that $s^t \neq s^{t-1}$, $(s^t)_{-i} = (s^{t-1})_{-i}$, and $u_i(s^t) > u_i(s^{t-1})$. The intuition behind this definition is that each deviator choose a better alternative. Γ is said to have the *Finite Improvement Property* (FIP) if and only if every improvement path is finite.

To establish our characterization, some technical limitations on games are required: A game $\Gamma = (A, S, u)$ is called *degenerate* iff there exist $i \in A$, $s_i, s_i' \in S_i$, $s_i \neq s_i'$, and $s_{-i} \in S_{-i}$ such that $u_i(s_i, s_{-i}) = u_i(s_i', s_{-i})$; otherwise, Γ is called *nondegenerate*.

Theorem 6.10 *Let Γ be a finite, nondegenerate game with utilities. Then, Γ has the FIP if and only Γ is an ordinal potential game.*

Proof:

($\Leftarrow$): Let $\Gamma = (A, S, u)$ be a finite game with ordinal potential function P, i.e., for all $i \in A$, $s_i, s_i' \in S_i$, $s_{-i} \in S_{-i}$,

$$u_i(s_i', s_{-i}) \geq u_i(s_i, s_{-i}) \Longleftrightarrow P(s_i, s_{-i}) \geq P(s_i', s_{-i}).$$

Let $\gamma = (s^0, s^1, s^2, \dots)$ be an improvement path, and let $(i_1, i_2, \dots)$ be the sequence of γ's deviators. Then, for all $t \in I, t > 0$, it holds that $u_{i_t}(s^t) > u_{i_t}(s^{t-1})$. Hence, $P(s^0) < P(s^1) < P(s^2) < \dots$. As S is a finite set, $\gamma = (s^0, s^1, s^2, \dots)$ is a finite sequence, i.e., $\|I\| < \infty$.

($\Rightarrow$): Let $\Gamma = (A, S, u)$ have the FIP. Define a binary relation $>$ on S:

$$s > s' \Longleftrightarrow_{\text{def}} s \neq s' \text{ and there is an improvement path from } s \text{ to } s'$$

Since Γ has the FIP, $>$ is a strict order relation on S, i.e., $>$ is irreflexive and transitive. Any finite strict order can be represented by a function: A set $Z \subseteq S$ is *represented* iff there is a mapping $Q : Z \to \mathbb{R}$ such that for all $s, s' \in Z$, $s > s'$ implies $Q(s) > Q(S')$. Let Z^* be a maximal, represented subset of S.

We show $Z^* = S$. To the contrary, assume there is an $x \in S$, $x \notin Z^*$. Then, there are three (possibly overlapping) cases:

1. There is no $z \in Z*$ such that $z > x$. Define an extension $Q' : Z^* \cup \{x\} \to \mathbb{R}$ by:

$$Q'(z) = \begin{cases} Q(z) & \text{if } z \in Z^* \\ \max\{\ Q(z) \mid z \in Z^*\ \} + 1 & \text{if } z = x \end{cases}$$

Q' represents $Z^* \cup \{x\}$ and, thus, contradicts to the maximality of Z^*.

2. There is no $z \in Z*$ such that $z < x$. Dually to the first case, define an extension $Q' : Z^* \cup \{x\} \to \mathbb{R}$ by:

$$Q'(z) = \begin{cases} Q(z) & \text{if } z \in Z^* \\ \min\{\ Q(z) \mid z \in Z^*\ \} - 1 & \text{if } z = x \end{cases}$$

   Q' represents $Z^* \cup \{x\}$ and, thus, contradicts to the maximality of Z^*.

3. For some $z, z' \in Z*$, it holds that $z > x > z'$. In this case, define an extension $Q' : Z^* \cup \{x\} \to \mathbb{R}$ by:

$$Q'(z) = \begin{cases} Q(z) & \text{if } z \in Z^* \\ \frac{1}{2}\left(\max\{\ Q(z) \mid z < x\ \} + \min\{\ Q(z) \mid z > x\ \}\right) & \text{if } z = x \end{cases}$$

   Q' represents $Z^* \cup \{x\}$ and, thus, contradicts to the maximality of Z^*.

Therefore, $Z^* = S$.

Let Q represent S. Then, Q is an ordinal potential function: Suppose $s_i, s_i' \in S_i$, $s_{-i} \in S_{-i}$. Then, $u_i(s_i, s_{-i}) \neq u_i(s_i', s_{-i})$ since Γ is nondegenerate. So, without loss of generality, $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$. Thus, $(s_i, s_{-i}) > (s_i', s_{-i})$. (Note there is an improvement path of length one.) Hence, $Q(s_i, s_{-i}) > Q(s_i', s_{-i})$.

This proves the theorem. ■

6.1.3 Congestion games

Congestion games have been introduced in economics by Robert W. Rosenthal in 1973. A scenario related to computer science is as follows. Suppose we are given the following interaction domain $\mathcal{I} \subseteq A \times A$, $A = \{\mathtt{A}, \mathtt{B}, \mathtt{C}, \mathtt{D}\}$:

...

Here, $\mathtt{A}, \mathtt{B}, \mathtt{C}$, and $\mathtt{D}$ are routers. Router $\mathtt{A}$ wishes to select to route to $\mathtt{C}$, and router $\mathtt{B}$ wishes to select a route to $\mathtt{D}$. If both routers use the same link then the congestion, or latency, increases according to cost function c_i. The routers aim at minimizing their costs.

We can analyze this scenario as a game with utilities

$$\Gamma =_{\text{def}} (\{\mathtt{A}, \mathtt{B}\}, \{\{1,2\},\{3,4\}\} \times \{\{1,3\},\{2,4\}\}, u)$$

where the utility function $u = (u_1, u_2)$ is given by the following bimatrix:

$$\begin{array}{c} \\ \{1,2\} \\ \{3,4\} \end{array} \begin{array}{c} \begin{array}{cc} \{1,3\} & \{2,4\} \end{array} \\ \begin{pmatrix} (c_1(2) + c_2(1), c_1(2) + c_3(1)) & (c_1(1) + c_2(2), c_2(2) + c_4(1)) \\ (c_3(2) + c_4(1), c_1(1) + c_3(2)) & (c_3(1) + c_4(2), c_2(1) + c_4(2)) \end{pmatrix} \end{array}$$

There are two simple, closed paths of length 4 in the game Γ. So, let p be the one in counter-clockwise direction starting with the upper left strategy profile corner, and let q be the one in clockwise direction also starting with the upper left strategy profile corner. It holds that

$$\begin{aligned} I(\Gamma,p) &= \underbrace{c_3(2)+c_4(1)-c_1(2)-c_2(1)}_{\texttt{A deviates}} + \overbrace{c_2(1)+c_4(2)-c_1(1)-c_3(2)}^{\texttt{B deviates}} + \\ &\quad + \underbrace{c_1(1)+c_2(2)-c_3(1)-c_4(2)}_{\texttt{A deviates}} + \overbrace{c_1(2)+c_3(1)-c_2(2)-c_4(1)}^{\texttt{B deviates}} \\ &= 0 \end{aligned}$$

Since $I(\Gamma,q) = -I(\Gamma,p) = 0$, we obtain from Theorem 6.9 that Γ is a potential game.

Definition 6.11 *A* congestion model *is a tuple* $(A, F, (S_i)_{i\in A}, (w_f)_{f\in F})$ *such that*

1. $A = \{1,\dots,n\}$ *is a non-empty, finite set of* agents *(routers),*
2. F *is a non-empty, finite set of* facilities *(links),*
3. $S_i \subseteq \mathcal{P}(F)$ *is a non-empty set of strategies (routes) for each agent* $i \in A$*, and*
4. $w_f : \{1,\dots,n\} \to \mathbb{R}$ *is a* cost *(wealth, latency)* function *for each facility* $f \in F$*; if* k *agents choose* f *then the cost for each agent is* $w_f(k)$*.*

Definition 6.12 *Let* $(A, F, (S_i)_{i\in A}, (w_f)_{f\in F})$ *be a congestion model. Then,* $\Gamma = (A, (S_i)_{i\in A}, u)$ *is called* congestion game *if and only if for all* $i \in A$*,* $s = (s_i, s_{-i}) \in S$*,*

$$u_i(s) = \sum_{f\in s_i} w_f(\sigma_f(s)),$$

where $\sigma_f(s) = \|\{i \in A | f \in s_i\}\|$*.*

Without proof we state the following theorem which shows that potential games and congestion games are essentially the same class of finite games.

Theorem 6.13 *1. Each congestion game is a potential game.*

2. Each potential game is isomorphic to a congestion game.

The proof of the first statement relies on the Rosenthal *potential*:

$$P(s) =_{\text{def}} \sum_{f\in\bigcup_{i\in A} s_i} \sum_{k=1}^{\sigma_f(s)} w_f(k)$$

6.2 Network potentials

To be filled in

Threshold models 7

Threshold models are a widely used class of models for behavioral attributes, that is, attributes on items. For instance, they have been used to model diffusion processes (contagion) of innovation, riots, rumors and diseases, strikes, voting (see, e.g, [16]), and furthermore in the context of neural networks (e.g., Hopfield networks).

Example: The following scenario was discussed by Granovetter in [16] to advocate the usage of threshold models in social sciences: Imagine there are 100 people milling around in a square–a potential riot situation. Now, assume that there are two slightly different threshold distribution among individuals:

1. There is one individual with threshold 0 (the instigator), one individual with threshold 1, and so on, and one individual with threshold 99. In a "domino" effect, the instigator breaks a window; this activates the person with threshold 1, and so on; finally, all 100 people have joined.
2. There is one individual with threshold 0, no individual with threshold 1, two individuals with threshold 2, one individual with threshold 3, and so on. That is, the crowds are essentially identical. Of course, the riots end with one rioter.

However, newspapers will likely react very differently:

1. "A crowd of radicals engaged in riotous behavior."
2. "A demented troublemaker broke a window while a group of solid citiziens looked on."

It is hazardous to infer individual dispositions from aggregate outcomes.

7.1 Boolean threshold models

An n-ary function $f : \{0,1\}^n \to \{0,1\}$ is a *threshold function* if and only if there are weights $w_1, \ldots, w_n \in \mathbb{R}_{\geq 0}$ and a threshold $\vartheta \in \mathbb{R}_{\geq 0}$ such that for all $z_1, \ldots, z_n \in \{0,1\}$,

$$f(z_1, \ldots, z_n) = 1 \iff \sum_{i=1}^{n} w_i \cdot z_i \geq \vartheta_i$$

Example: We discuss several functions.

- The standard case of a threshold function is the n-ary *majority function* which is specified via weights $w_1 = \cdots = w_n = 1$ and threshold $\vartheta = \lceil \frac{n+1}{2} \rceil$.

- Which binary functions are threshold functions?
  - AND is a threshold function: $w_1 = w_2 = 1$, $\vartheta = 2$:

x_1	x_2	$x_1 \wedge x_2$
0	0	0
0	1	0
1	0	0
1	1	1

  - OR is a threshold function: $w_1 = w_2 = 1$, $\vartheta = 1$:

x_1	x_2	$x_1 \vee x_2$
0	0	0
0	1	1
1	0	1
1	1	1

  - XOR is not a threshold function:

x_1	x_2	$x_1 \oplus x_2$
0	0	0
0	1	1
1	0	1
1	1	0

- The ternary function $f : \{0,1\}^3 \to \{0,1\} : (x_1, x_2, x_3) \mapsto (x_1 \wedge x_2) \vee x_3$ is a threshold function, e.g., via $w_1 = 2, w_2 = 2, w_3 = 5$, and $\vartheta = 4$:

x_1	x_2	x_3	$f(x_1, x_2, x_3)$
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

  Observe that f is the disjunction of two threshold functions.

- In light of the preceding example, is $(x_1 \wedge x_2) \vee (x_3 \wedge x_4)$ then a threshold function as well? No, it is not. Assume it is. Then, there are weights w_1, w_2, w_3, w_4 such that

$$(x_1 \wedge x_2) \vee (x_3 \wedge x_4) \equiv 1 \iff w_1 \cdot x_1 + w_2 \cdot x_2 + w_3 \cdot x_3 + w_4 \cdot x_4 \geq 1.$$

So, the following inequalities are true:

$w_1 + w_2 \geq 1$	since $(1, 1, 0, 0)$ is a satisfying assignment
$w_3 + w_4 \geq 1$	since $(0, 0, 1, 1)$ is a satisfying assignment
$w_2 + w_3 < 1$	since $(0, 1, 1, 0)$ is not a satisfying assignment
$w_1 + w_4 < 1$	since $(1, 0, 0, 1)$ is not a satisfying assignment

Consequently, $2 \leq w_1 + w_2 + w_3 + w_4 < 2$. A contradiction.

7.2 Equilibria in threshold models

We want to find fixed points in networks over threshold functions. In fact, we prove that fixed points always exist in networks where local transitions belong to a larger class of functions.

A function $f : \{0,1\}^n \to \{0,1\}^k$ is said to be *monotone* if and only if for all $x, y \in \{0,1\}^n$, $x \leq y$ implies $f(x) \leq f(y)$. The less-than-or-equal relation is defined to be the vector-ordering.

Proposition 7.1 *Each threshold function is monotone.*

Proof: Suppose $x = (x_1, \ldots, x_n) \leq (y_1, \ldots, y_n) = y$, i.e., $x_i \leq y_i$ for all $i \in \{1, \ldots, n\}$, and $f(x) = 1$. Since the weights $w_1, \ldots, w_n$ are non-negative, we obtain

$$\vartheta \leq \sum_{i=1}^{n} w_i \cdot x_i \leq \sum_{i=1}^{n} w_i \cdot y_i.$$

Thus, $f(y) = 1$. Therefore, f is a monotone function. This proves the proposition. ■

Note that in the examples above, the last function is monotone, though not a threshold function.

Proposition 7.2 *Let $L = \{f_1, \ldots, f_m\}$ be a set of local transitions on an item set A such that all f_i are threshold functions. Then, $\mathbf{F}_L[U]$ is monotone (with respect to the vector-ordering) for all $\emptyset \neq U \subseteq A$.*

Proof: Each component of $\mathbf{F}_L[U]$ is a threshold function. That is, each component of $\mathbf{F}_L[U]$ is monotone. Hence, $\mathbf{F}_L[U]$ is monotone with respect to the vector-ordering. This shows the proposition. ■

Theorem 7.3 *Let $L = \{f_1, \ldots, f_n\}$ be a set of local transition functions on an item set A. Let α be any schedule on A. Then, the global network map $\mathbf{F}_{(L,\alpha)}$ has a fixed point.*

Proof: We give an algorithm for finding a fixed point of $\mathbf{F}_{(L,\alpha)}$ that, essentially, simulates the global network map along a specific orbit:

```
[1] x := (0, 0, ..., 0)
[2] repeat
[3]     y := x
[4]     x := F_(L,α)(y)
[5] until x = y
```

If the procedure terminates then a fixed point is found. To show the termination property, observe that we construct an ascending chain

$$x^{(0)} \leq x^{(1)} \leq x^{(2)} \leq x^{(3)} \leq \cdots \leq x^{(k)} \leq \ldots,$$

where $x^{(0)} =_{\text{def}} (0, 0, \ldots, 0)$ and $x^{(k)}$ is the configuration assigned to x in the fourth line of the algorithm above. Indeed, this is easily seen by induction:

- *base of induction* $n = 1$: It holds that

  $$x^{(0)} \leq \mathbf{F}_{(M,\alpha)}(x^{(0)}) = x^{(1)}$$

  since $x^{(0)}$ is a bottom element in the poset $(\{0,1\}^n, \leq)$ and since $\mathbf{F}_{(M,\alpha)}$ is monotone by Proposition 7.2.

- *inductive step* $n > 1$: Using the monotonicity of $\mathbf{F}_{(M,\alpha)}$, we obtain

  $$\begin{aligned} x^{(n)} &= \mathbf{F}_{(M,\alpha)}(x^{(n-1)}) && \text{(line (4) of the algorithm)} \\ &\leq \mathbf{F}_{(M,\alpha)}(x^{(n)}) && \text{(by inductive assumption and Proposition 7.2)} \\ &= x^{(n+1)} && \text{(line (4) of the algorithm)} \end{aligned}$$

Since $\{0,1\}^n$ is a finite set, the chain si finite. Thus, the procedure terminates. Therefore, there a fixed point for (M, α). This proves the theorem. ∎

Note that, in the proof above, there is a dual procedure starting at $(1, 1, \ldots, 1)$.

Models of opinion dynamics 8

8.1 Attitudes and attitude change

An *attitude* is a positive, negative, or mixed reaction to a person, object, or idea. Attitudes are (often) measured using (multi-item) questionnaires known as *attitude scales*, e.g., Likert scales. Note that attitude scales are based on the principle of forced choices. An *opinion* is the result of selecting a value from an attribute range D. That is, we consider an opinion $o_i \in D$ of actor $i \in A$ is the state of i in the population A.

In the forthcoming, we consider attitude change by persuasive communication. Among the many models developped in social psychology, a standard model is the Elaboration Likelihood Model (ELM), a dual-process model by Petty and Cacioppo. It makes an antagonistic distinction between two ways to persuasion depending on the so-called *need for cognition* of an audience receiving a message from a source:

- *central route* to persuasion: actor thinks carefully about a communication and is influenced by the strength of its arguments.
- *peripheral route* to persuasion: actor does not think carefully about a communication and is influenced by superficial cues.

Models of opinion dynamics implement mechanisms of interdependent influence by persuasive communication affecting opinion changes, and by this presumably attitude changes, of the actors.

8.2 The Friedkin-Johnsen model

The Friedkin-Johnsen model is an opinion dynamics model based on social influence, first published in 1990. It is representative for the class of linear models.

In general, the model is given for a sequence of opinion vectors $o^{(k)} = (o_1^{(k)}, o_2^{(k)}, \ldots, o_n^{(k)})$ for $k \in \mathbb{N}$:

$o^{(0)} = X_1 B_1$ where

$X_1 \in \mathbb{R}^{n \times k}$ represents weighted influence of k exogenous variables
$B_1 \in \mathbb{R}^{n \times k}$ represents values of j exogenous variables

$$o^{(k)} = \alpha_k W_k o^{(k-1)} + \beta_k X_k B_k \text{ where}$$

$X_k \in \mathbb{R}^{n\times k}, B_l \in \mathbb{R}^{n\times k}$ have the same meaning as above
$W_k \in \mathbb{R}^{n\times n}$ represents the network of influence
$\alpha_k \in \mathbb{R}^{n\times n}$ represents a weight on the endogenous conditions
$\beta_k \in \mathbb{R}^{n\times n}$ represents a weight on the exogenous conditions

So, in the general model, for each time set we have different influences of the variables and in the networks. This makes the model almost unanalyzable.

The basic model assumes that the description is static: $X_1 = X_2 = \cdots = X$, $B_1 = B_2 = \cdots = B$, $W_1 = W_2 = \cdots = W$, $\alpha_1 = \alpha_2 = \cdots = \alpha$, $\beta_1 = \beta_2 = \cdots = \beta$. The basic model, in fact, defines an iterated map on some domain $D \subseteq \mathbb{R}$:

$$F : D^n \to D^n : x \mapsto \alpha W x + \beta X B$$

Let us consider the orbit of $x^{(0)} = XB$ under F:

$$\begin{aligned}
x^{(0)} &= XB \\
x^{(1)} &= F(x^{(0)}) = \alpha W X B + \beta X B = (\alpha W + \beta I) X B \\
x^{(2)} &= F(x^{(1)}) = \alpha W (\alpha W + \beta I) X B + \beta X B = (\alpha^2 W^2 + \alpha\beta W + \beta I) X B \\
x^{(3)} &= F(x^{(2)}) = \alpha W (\alpha^2 W^2 + \alpha\beta W + \beta I) X B + \beta X B \\
&\qquad = (\alpha^3 W^3 + \alpha^2 \beta W^2 + \alpha\beta W + \beta I) X B
\end{aligned}$$

An easy inductive argument shows that for the k-th iteration F^k:

$$x^{(k)} = \alpha^k W^k X B + \left(\sum_{t=0}^{k-1} \alpha^t W^t \right) \beta X B$$

We obtain the following by using results from linear algebra:

- $\lim_{k\to\infty} \alpha^k W^k X B = 0$ if $|\alpha| < 1$ and there is an m such that $|W^k| \leq m \cdot 1_{n\times n}$ for all $k \in \mathbb{N}$ (note that $1_{n\times n}$ is the all-one $n \times n$ matrix).
- $\sum_{t=0}^{k-1} \alpha^t W^t = (I - \alpha W)^{-1}$ if α^{-1} is not an eigenvalue of W

Overall, if all conditions are satisfied, we obtain $\lim_{k\to\infty} x^{(k)} = (I - \alpha W)^{-1} \beta X B$. The opinion change process converges and reaches an equilibrium.

8.3 The Relative Agreement model

The Relative Agreement model has been introduced in the paper

> Guillaume Deffuant, Frédéric Amblard, Gérard Weisbuch, Thierry Faure: How can extremism prevail? A study on the relative agreement interaction model. *Journal of Artificial Societies and Social Simulation*, 5(4), 2002.

in order look for network parameters that allow extreme opinions to dominate eventually within a human population. The model is representative for a physics-oriented approach to complex networks: (a) methodologically, it employs *agent-based modelling*; (b) it explains a complex phenomenon in a *stylized*, metaphorical fashion. Apart from the methodological perspective, the concrete, original research motivation for the study presented lies in the influence "green" farmers have attained in the farming population.

We consider the following formal scenario: A population of n agents is given. An agent i is characterized by two variables:

- opinion $x_i \in [-1, 1]$
- uncertainty $u_i \in [0, 1]$

Thus, the actual opinion of the agents ranges in her opinion segment

$$S_i =_{\text{def}} [x_i - u_i, x_i + u_i],$$

the size of which is $(x_i + u_i) - (x_i - u_i) = 2u_i$.

We suppose a directed model of influence where any two agents use a communication channel. Agent i locally communicates to agent j over her communication channel, possibly causing changes in opinion and uncertainty of agent j. In this situation, agent i is the influencer of agent j and agent j is the influenced of agent i.

The effect of a communicative influence is given by an update rule which is assumed to be the same for all interaction pairs of agents. Figure 8.1 describes a situation of an interaction pair (i, j).

The update rule is based on the agreement along the opinion segments of agents i and j, i.e.,

$$h_{ij} - (2u_i - h_{ij}) = 2(h_{ij} - u_i),$$

in relation to the uncertainty of the influencer

$$\frac{2(h_{ij} - u_i)}{2u_i} = \frac{h_{ij}}{u_i} - 1.$$

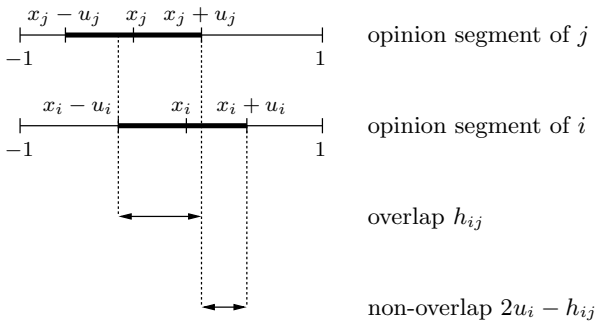


Figure 8.1: The Relative Agreement model

The formal specification of the update rule is given by defining local transitions:

$$x_j \leftarrow x_j + \begin{cases} \mu\left(\frac{h_{ij}}{u_i} - 1\right)(x_i - x_j) & \text{if } h_{ij} \geq u_i \\ 0 & \text{otherwise} \end{cases}$$

$$u_j \leftarrow u_j + \begin{cases} \mu\left(\frac{h_{ij}}{u_i} - 1\right)(u_i - u_j) & \text{if } h_{ij} \geq u_i \\ 0 & \text{otherwise} \end{cases}$$

Here, μ is some decay constant, $0 < \mu < 1$.

Proposition 8.1 *Let an interaction pair (i, j) be given. Let h_{ij} denote the overlap of the opinion segments of the actors i and j before interaction, and let h'_{ij} denote the overlap of the opinion segments of the actors i and j after interaction. Then, $h_{ij} \leq h'_{ij}$.*

Proof: Let (x_i, u_i) be the opinion/uncertainty pair of actor i, let (x_j, u_j) be the opinion/uncertainty pair of actor j. According to our update rule, if $h_{ij} \leq u_i$ then there are no changes, neither in the opinions nor in the uncertainties of both actors. That is, $h'_{ij} = h_{ij}$. So, let $h_{ij} > u_i$. Let x'_j denote actor j's opinion after interaction, and let u'_j denote actor j's uncertainty after interction. More specifically, we have

$$\begin{aligned} x'_j &= \left(1 - \mu\left(\frac{h_{ij}}{u_i} - 1\right)\right) x_j + \mu\left(\frac{h_{ij}}{u_i} - 1\right) x_i, \\ u'_j &= \left(1 - \mu\left(\frac{h_{ij}}{u_i} - 1\right)\right) u_j + \mu\left(\frac{h_{ij}}{u_i} - 1\right) u_i. \end{aligned}$$

The overlap h'_{ij} is given by

$$h'_{ij} = \min(x_i + u_i, x'_j + u'_j) - \max(x_i - u_i, x'_j - u'_j).$$

Note that $h_{ij} \leq 2u_i$. Thus, the update rules define convex combinations. By linearity, we easily examine the following cases:

1. Suppose $x_i + u_i \leq x_j + u_j$ and $x_i - u_i \geq x_j - u_j$. Therefore,

$$\begin{aligned} x_i + u_i &\leq x'_j + u'_j \leq x_j + u_j, \\ x_j - u_j &\leq x'_j - u'_j \leq x_i - u_i. \end{aligned}$$

   We obtain

$$\begin{aligned} h_{ij} &= x_i + u_i - (x_i - u_i), \\ h'_{ij} &= (x_i + u_i) - (x_i - u_i) = h_{ij} \end{aligned}$$

2. Suppose $x_i + u_i \leq x_j + u_j$ and $x_i - u_i \leq x_j - u_j$. Therefore,

$$\begin{aligned} x_i + u_i &\leq x'_j + u'_j \leq x_j + u_j, \\ x_i - u_i &\leq x'_j - u'_j \leq x_j - u_j. \end{aligned}$$

   We obtain

$$\begin{aligned} h_{ij} &= x_i + u_i - (x_j - u_j), \\ h'_{ij} &= (x_i + u_i) - (x'_j - u'_j) \geq (x_i + u_i) - (x_j - u_j) = h_{ij} \end{aligned}$$

3. Suppose $x_i + u_i \geq x_j + u_j$ and $x_i - u_i \leq x_j - u_j$. Therefore,

$$\begin{aligned} x_j + u_j &\leq x'_j + u'_j \leq x_i + u_i, \\ x_i - u_i &\leq x'_j - u'_j \leq x_j - u_j. \end{aligned}$$

   We obtain

$$\begin{aligned} h_{ij} &= x_j + u_j - (x_j - u_j), \\ h'_{ij} &= (x'_j + u'_j) - (x'_j - u'_j) \geq (x_i + u_i) - (x_j - u_j) = h_{ij} \end{aligned}$$

4. Suppose $x_i + u_i \geq x_j + u_j$ and $x_i - u_i \geq x_j - u_j$. Therefore,

$$\begin{aligned} x_j + u_j &\leq x'_j + u'_j \leq x_i + u_i, \\ x_j - u_j &\leq x'_j - u'_j \leq x_i - u_i. \end{aligned}$$

   We obtain

$$\begin{aligned} h_{ij} &= x_j + u_j - (x_i - u_i), \\ h'_{ij} &= (x'_j + u'_j) - (x_i - u_i) \geq (x_j + u_j) - (x_i - u_i) = h_{ij} \end{aligned}$$

This proves the proposition. ■

Proposition 8.2 *Let the interaction pair (i,j) be given. For $k \in \mathbb{N}$, let $x_j^{(k)}$ be actor j's opinion after the k-th round of the directed interaction (i,j), and let $u_j^{(k)}$ be actor j's uncertainty after the k-th round of the directed interaction (i,j). Then,*

$$\lim_{k\to\infty} x_j^{(k)} = x_i, \qquad \lim_{k\to\infty} u_j^{(k)} = u_i.$$

Proof: We only prove the convergence in opinions. Since $h_{ij} \leq 2u_i$, we obtain as an upper bound on the opinion $x_j^{(k)}$ for $k \in \mathbb{N}_+$

$$x_j^{(k)} \leq (1-\mu)x_j^{(k-1)} + \mu x_i,$$

and furthermore, by induction,

$$x_j^{(k)} \leq (1-\mu)^k x_j + \left(1-(1-\mu)^k\right) x_i.$$

Hence, $\lim_{k\to\infty} x_j^{(k)} \leq x_i$. For the lower bound, we write

$$x_j^{(k)} = (1-\mu A_{k-1})x_j^{(k-1)} + \mu A_{k-1} x_i,$$

where $A_k = \frac{h_{ij}^{(k)}}{u_i} - 1$ is the relative agreement after the k-th interaction. By Proposition 8.1, it holds that $A_k \leq A_{k+1}$ for all $k \in \mathbb{N}$. Thus, we can estimate

$$x_j^{(k)} \geq (1-\mu A_0)x_j^{(k-1)} + \mu A_0 x_i,$$

and, again by induction,

$$x_j^{(k)} \geq (1-\mu A_0)^k x_j + \left(1-(1-\mu A_0)^k\right) x_i,$$

Hence, $\lim_{k\to\infty} x_j^{(k)} \geq x_i$. This proves the proposition. ■

In general populations of actors, it is not clear at all whether there is any convergence to a "stable" opinion/uncertainty pattern over several time steps. If unambiguous convergence is reachable, there are three important cases: convergence to the opinion poles, either positive or negative, or convergence to the middle. We are interest in studying convergence to extreme opinions.

Extremists are people with extreme opinions, i.e., opinions close to the boundaries measured by -1 and 1. Furthermore, the model of extremists within a population is based on two observations which are claimed to possess a certain anectodal evidence [7]:

1. "... people who have extreme opinions tend to be more convinced,"

2. "... people who have moderate initial opinions, often express a lack of knowledge (and uncertainty)."

A simplification of these observations can be incorporated into the Relative Agreement model as follows. Let u_e be the uncertainties of the extremists, supposed to be small and the same for all extremists. Let u be the (identical) uncertainty of the moderate. According to our observations, it holds that $u > u_e$. Then, the population can be initially decomposed into three classes corresponding to their opinion/uncertainty pairs:

1. *positive extremists*: $x_i \approx 1$, $u_i = u_e$
2. *negative extremists*: $x_i \approx -1$, $u_i = u_e$
3. *moderates*: $x_i \approx 0$, $u_i = u$

Let p_e denote the fraction of extremists, either positive or negative, in the population. Depending on the fraction of actors belonging to these classes, an extremism bias can be defined. Let p_+ be the fraction of positive extremists, and let p_- be the fraction of negative extremists. Then, the *extremism bias* δ is given as

$$\delta = \frac{p_+ - p_-}{p_+ + p_-}$$

The simulation works in two phases:

1. For initalization, (a) choose n opinions uniformly at random from $[-1, 1]$ and set all n uncertainties to u, (b) for the $p_+ \cdot n$ most positive opinions and $p_- \cdot n$ most negative opinons, the uncertainties are set to u_e.
2. Iteratively choose a pair (i, j) of agents and let agent i exert influence on agent j according to the specified update rule.

The stylized simulation results can be divided into three stable scenarios: central clustering, bipolarization, single polarization. The following figures show diagram schemes for each of the three scenarios together with parameter settings such that the described behavior can be observed. The x-axis codes for time, i.e., number of iterations per actor, and the y-axis codes for opinions. A trajectory of an actor's opinion over the course of time runs inside the region bounded by the drawn curves. Common parameters for all figures (and the simulations) are $n = 200$, $\mu = 0.5$, $\delta = 0$, and $u_e = 0.1$. The initial uncertainty parameter u is increased from figure to figure.

In Figure 8.2, the initial uncertainty of the moderates is $u = 0.4$. It is an example of central convergence. The majority of the moderate actors are not attracted by the extreme opinions.

In Figure 8.3, the initial uncertainty of the moderates is $u = 1.2$. It is an example of convergence to both extremes. The initially moderate actors split and become extremists.

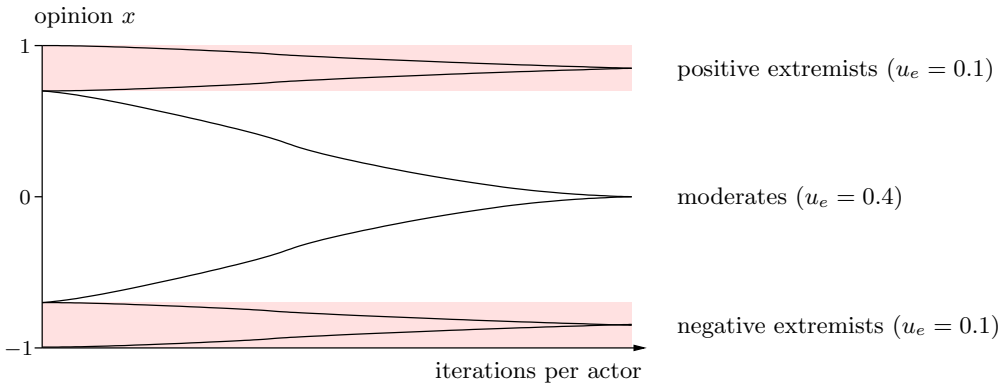


Figure 8.2: Scheme of central convergence.

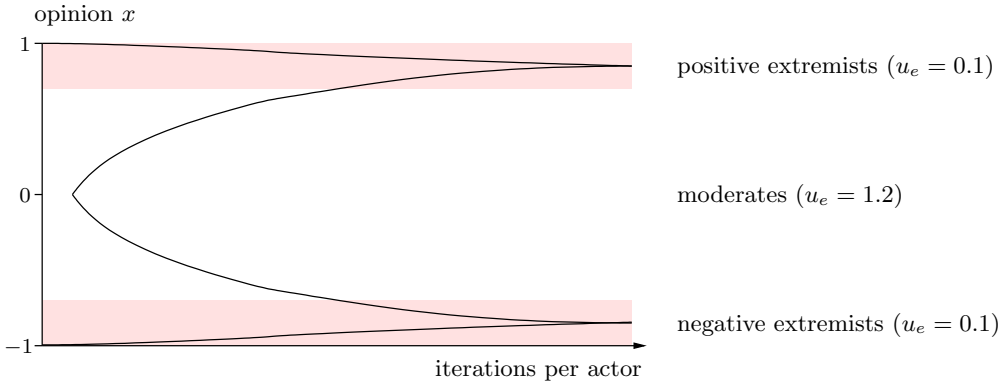


Figure 8.3: Scheme of bipolarization.

In Figure 8.4, the initial uncertainty of the moderates is $u = 1.4$. It is an example of convergence to one single extreme. In this case, the majority of the population is attracted by one of the extremes. This behavior can take place even when the number of initial extremissts is the same at both extremes, which is claimed to have been a priori unexpected.

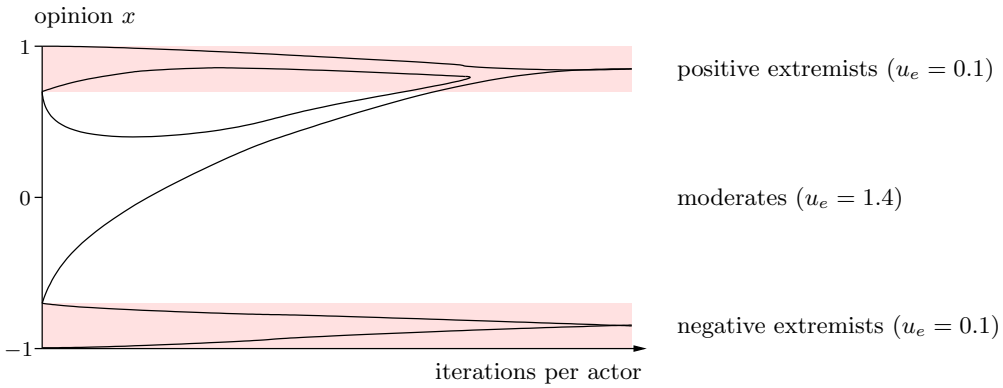


Figure 8.4: Scheme of single polarization.

Mathematical tools A

In this chapter we discuss relevant terminology and notation for sets, relations, and graphs, some fundamental algorithms, and a few other mathematical preliminaries.

A.1 Sets and relations

We denote the set of integers by $\mathbb{Z}$, the set of non-negative integers by $\mathbb{N}$, and the set of positive integers by $\mathbb{N}_+$. $\mathbb{Z}_2$ denotes the Galois field GF[2].

Sets

The empty set is denoted by $\emptyset$. For an arbitrary set A, $\mathcal{P}(A)$ denotes the power set of A, i.e., the family of all subsets of A, and $\mathcal{P}_+(A)$ denotes the set $\mathcal{P}(A) \setminus \{\emptyset\}$. For an arbitrary finite set A, its cardinality is denoted by $\|A\|$. Let A and B be any sets. Then $A \setminus B$ denotes the difference of A with B, i.e., the set of all elements that are in A but not in B. $A \times B$ denotes the cartesian product, i.e, the set of all pairs (a, b) with $a \in A$ and $b \in B$. For $m \in \mathbb{N}_+$, define $A^m =_{\text{def}} \underbrace{A \times \cdots \times A}_{m \text{ times}}$. Let M be any fixed basic set. For a set $A \subseteq M$, its complement in the basic set M is denoted by $\overline{A}$, i.e., $\overline{A} = M \setminus A$. A multiset A is allowed to contain elements many times. The multiplicity of an element x in a multiset A is the number of occurrences of x in A. The cardinality of a multiset A is also denoted by $\|A\|$.

Functions

Let M and M' be any sets, and let $f : M \to M'$ by any function. The domain of f which we denote by D_f is the set of all $x \in M$ such that $f(x)$ is defined. A function f is total if the domain of f is M. For a set $A \subseteq D_f$, let $f(A) = \{f(x) \mid x \in A\}$ denote the image of A under f. In particular, the range of f which is denoted by R_f is the set $f(D_f)$. For a set $A \subseteq M$, the restriction of a total function f to A is denoted by $f[A]$. The inverse of f is denoted by f^{-1}, i.e, $f^{-1} : M' \to \mathcal{P}(M)$ such that for all $y \in M'$, $f^{-1}(y) = \{x \in M \mid f(x) = y\}$. If $f^{-1}(y)$ is at most a singleton then we omit the braces. The pre-image of A under f is the set $f^{-1}(A) = \{x \in M \mid f(x) \in A\}$.

We use two notations for composition of functions. If f and f' are functions with $f : M \to M'$ and $f' : M' \to M''$, then $(f' \circ f)$ is the function mapping from M to

M'' which is defined for all $x \in M$ as $(f' \circ f)(x) \quad =_{\text{def}} \quad f'(f(x))$. In contrast, we use $f \cdot f'$ to denote $f' \circ f$.

A function $f : M \to M'$ is bijective if f is surjective, i.e., $R_f = M'$ and injective, i.e., for all $y \in R_f$, $f^{-1}(y)$ is a singleton. Suppose $M' = M$ and M is finite. In this case a bijective function f is a permutation. Suppose $M = \{1, 2, \ldots, n\}$. A *cycle* $(i_1\ i_2\ \ldots\ i_k)$ of length k of the permutation $\pi : M \to M$ is a sequence $(i_1, i_2, \ldots, i_k)$ such that $\pi(i_j) = i_{j+1}$ for $1 \leq j < k$ and $\pi(i_k) = i_1$. Each permutation allows a decomposition into cycles.

Orders

In more detail the following can be found in any textbook (e.g., [17, 6]) about theory of orders and lattices.

Let P be any set. A *partial order* on P (or order, for short) is a binary relation $\leq$ on P that is reflexive, antisymmetric, and transitive. The set P equipped with a partial order $\leq$ is said to be a *partially ordered set* (for short, *poset*). Usually, we talk about the poset P. Where it is necessary we write $(P, \leq)$ to specify the order. A poset P is a *chain* if for all $x, y \in P$ it holds that $x \leq y$ or $y \leq x$ (i.e., any two elements are comparable with respect to $\leq$). Such an order is also called a *total order*. A poset P is an *antichain* if for all $x, y \in P$ it holds that $x \leq y$ implies that $x = y$ (i.e., no two elements are comparable with respect to $\leq$).

We consider $\mathbb{N}$ to be ordered by standard total order on the natural numbers. If a set A is partially ordered by $\leq$ then A^m can be considered to be ordered by the *vector-ordering*, i.e., $(x_1, \ldots, x_m) \leq (y_1, \ldots, y_m)$ if and only if for all $i \in \{1, \ldots, m\}$, $x_i \leq y_i$.

An important tool for representing posets is the *covering relation* $\prec$. Let P be a poset and let $x, y \in P$. We say that x is covered by y (or y covers x), and write $x \prec y$, if $x < y$ and $x \leq z < y$ implies that $x = z$. The latter condition is demanding that there be no element z of P with $x < z < y$. A finite poset P can be drawn in a diagram consisting of points (representing the elements of P) and interconnecting lines (indicating the covering relation) as follows: To each element x in P associate a point $P(x)$ in the picture which is above all points $P(y)$ associated to elements y less than x, and connect points $P(x)$ and $P(y)$ by a line if and only if $x \prec y$. A poset can have different representation by diagrams.

Let P and P' be posets. A map $\varphi : P \to P'$ is said to be *monotone* (or order-preserving) if $x \leq y$ in P implies $\varphi(x) \leq \varphi(y)$ in P'. We say that φ is an *(order-)isomorphism* if φ is monotone, injective, and surjective. Two posets P and P' are *isomorphic*, in symbols $P \cong P'$, if there exists an isomorphism $\varphi : P \to P'$. Isomorphic poset shall be considered to be not essentially different: Two finite posets are isomorphic if and only if they can be drawn with identical diagrams.

Words

Sometimes we make no difference between m-tuples $(x_1, \ldots, x_m)$ over a finite set M and words $x_1 \ldots x_m$ of length m over M. Such finite sets are called alphabets. Let Σ be a finite alphabet. Σ^* is the set of all finite words that can be built with letters from Σ. For $x, y \in \Sigma^*$, $x \cdot y$ (or xy for short) denotes the concatenation of x and y. The empty word is denoted by ε. For a word $x \in \Sigma^*$, $|x|$ denotes the length of x. For $n \in \mathbb{N}$, Σ^n is the set of all words $x \in \Sigma^*$ such that with $|x| = n$. For a word $x = x_1 \ldots x_n \in \Sigma^*$ any word $x_1 \ldots x_k$ such that $k \leq n$ is called a prefix of x. We use regular expressions to describe subsets of Σ^* (see, e.g., [23]).

A.2 Graph theory

A graph $G = (V, E)$ consists of a set V of vertices and a set E of edges joining pairs of vertices. The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. The cardinality of V is usually denoted by n, the cardinality of E by m. If two vertices are joined by an edge, they are adjacent and we call them *neighbors*. Graphs can be undirected und directed. In undirected graphs, the order in which vertices are joined is irrelevant. An undirected edge joining vertices $u, v \in V$ is denoted by $\{u, v\}$. In directed graphs, each directed edge has an *origin* and a *destination*. An edge with origin $u \in V$ and destination $v \in V$ is represented by an ordered pair (u, v). For a directed graph $G = (V, E)$, the *underlying undirected graph* is the undirected graph with vertex set V that has an undirected edge between two vertices $u, v \in V$ if (u, v) or (v, u) is in E.

Multigraphs

In both undirected and directed graphs, we may allow the edge set E to contain the same edge several times, i.e., E can be a multiset. If an edge occurs several times in E, the copies of that edge are called *parallel edges*. Graphs with parallel edges are also called *multigraphs*. A graph is called *simple*, if each of its edges in contained in E only once, i.e., if the graph does not have parallel edges. An edge joining a vertex to itself, is called a *loop*. A graph is called *loopless* if it has no loops. In general, we assume all graphs to be loopless unless specified otherwise.

Degrees

The *degree* of a vertex v in an undirected graph $G = (V, E)$, denoted by d_v, is the number of edges in E joining v. If G is a multigraph, parallel edges are counted according to their multiplicity in E. The set of neighbors of v is denoted by $N(v)$. $N^0(v)$ denotes the vertex set $N(v) \cup \{v\}$. If the graph under consideration is not clear from the context, these notations can be augmented by specifying the graph as an index. For example, $N_G(v)$ denotes the neighborhood of v in G.

Subgraphs

A graph $G' = (V', E')$ is a *subgraph* of the graph $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. Sometimes we denote this by $G' \subseteq G$. It is a *(vertex-)induced subgraph* if E' contains all edges $e \in E$ that join vertices in V'. The induced subgraph of $G = (V, E)$ with vertex set $V' \subseteq V$ is denoted by $G[V']$. The *(edge-)induced subgraph* with edge set $E' \subseteq E$, denoted by $G[E']$, is the subgraph $G' = (V', E')$ of G, where V' is the set of all vertices in V that are joined by at least one edge in E'.

Walks, paths, and cycles

A *walk* from x_0 to x_k in a graph $G = (V, E)$ is a sequence $x_0, e_1, x_1, e_2, x_2, \ldots, x_{k-1}, e_k, x_k$ alternating between vertices and edges of G, where $e_i = \{x_{i-1}, x_i\}$ in the undirected case and $e_i = (x_{i-1}, x_i)$ in the directed case. The length of a walk is the number of edges on the walk. As shorthands we use $(x_0, x_1, \ldots, x_k)$ and $(e_1, e_2, \ldots, e_k)$ to denote a walk. The walk is called a *path* if $x_i \neq x_j$ for $i \neq j$. A walk with $x_0 = x_k$ is called a *cycle* if $e_i \neq e_j$ for $i \neq j$. A cycle is a *simple cycle* if $x_i \neq x_k$ for $0 \leq i < j \leq k - 1$.

Special graphs

A *tree* is a connected (for a definition see below) undirected graph not containing a cycle. An undirected graph $G = (V, E)$ is called *complete* if it contains all possible pairs of vertices as edges. A complete graph with n vertices is denoted by K^n. A K^n is called a *clique*. A K^2 is a graph of two vertices with one edge joining them. A K^3 is also called a *triangle* or *triad*. A graph without edges is called *empty*. An *independent set* within a graph $G = (V, E)$ is a vertex set $U \subseteq V$ such that $G[U]$ is empty. A graph $G = (V, E)$ is called *bipartite* if there are independent vertex sets $V_1, V_2 \subseteq V$ such that V_1 and V_2 are disjoint and $V_1 \cup V_2 = V$. We denote by $E(V_1, V_2)$ the set of edges joining vertices from V_1 with vertices from V_2. If $E(V_1, V_2) = V_1 \times V_2$ then G is called a *complete bipartite graph*. Such a graph is denoted by K_{n_1,n_2} if V_1 consists of n_1 vertices and V_2 of n_2 vertices. A $K_{1,n}$ is also called a *star*. For two graphs $G = (V, E)$ and $G' = (V', E')$ we denote by $G \oplus G'$ the graph consisting of the disjoint union of the graphs G and G'.

Graph classes

Two graphs $G = (V, E)$ and $G' = (V', E')$ are *isomorphic*, denoted by $G \simeq G'$, if there is a bijective mapping $\varphi : V \to V'$ such that for all vertices $u, v \in V$ the following is true: in the case that G and G' are directed graphs it holds that $(u, v) \in E \Leftrightarrow (\varphi(u), \varphi(v)) \in E'$, and in the case that G and G' are undirected graphs it holds that $\{u, v\} \in E \Leftrightarrow \{\varphi(u), \varphi(v)\} \in E'$. A set of graphs is called a *graph class* if for each graph G in the class all graphs isomorphic to G belong to the class as well.

A.3 Algorithmics

Most results of this work relate to algorithms. In the following we mention essential problems and concepts which are needed more than once.

For two functions $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ we say that f is in $O(g)$ if there are constant $n_0, c \in \mathbb{N}_+$ such that for all $n \geq n_0$, $f(n) \leq c \cdot g(n)$. We say that f is in $\Omega(g)$ if g is in $O(f)$. We say that f is in $\Theta(g)$ if f is in $O(g) \cap \Omega(g)$.

Connected components

An undirected graph $G = (V, E)$ is *connected* if every vertex can be reached from every other vertex, i.e., if there is a path from every vertex to every other vertex. A graph consisting of a single vertex is also taken to be connected. Graphs that are not connected are called *disconnected*. For a given undirected graph $G = (V, E)$, a *connected component* of G is an induced subgraphs $G' = (V', E')$ that is connected and maximal, i.e., there is no connected subgraph $G'' = (V'', E'')$ such that $V'' \supset V'$. Checking whether a graph is connected and finding all its connected components can be done in time $O(n + m)$ using depth-first search or breadth-first search.

A directed graph $G = (V, E)$ is *strongly connected* if there is a directed path from every vertex to every other vertex. A *strongly connected component* of a directed graph G is an induced subgraph that is strongly connected and maximal. The strongly connected components of a directed graph can be computed in time $O(n + m)$ using a depth-first search.

NP-*completeness*

It is important to consider the running-time of an algorithm for a given problem. Usually, one wants to give an upper bound on the running time of the algorithm for inputs of a certain size. If the running-time of an algorithm is $O(n^k)$ for some $k \in \mathbb{N}$ and for inputs of size n, we say that the algorithm runs in polynomial time. For graph problems, the running-time is usually specified as a function of n and m, the number of vertices and edges of the graph, respectively. For many problems, however, no polynomial-time algorithm has been discovered. Although one cannot rule out the possible existence of polynomial-time algorithms for such problems, the theory of NP-completeness provides means to give evidence for the computational intractability of a problem.

A decision problem is in the complexity class NP if there is a nondeterministic Turing machine that solves the problem in polynomial time. That is to say that the answer to a problem instance is “yes” if there exists a solution in the set of all possible solutions to the instance which is of polynomial size. Moreover, the test whether a potential solution is an actual solution must be performed in polynomial time. Note that a decision problem

is usually considered to consist of the set of the "yes"-instances. A decision problem is NP-hard if every problem in NP can be reduced to it via a polynomial-time many-one reduction. (A polynomial-time many-one reduction from a set A to a set B is a function computable in polynomial time such that for all instances x, $x \in A \Leftrightarrow f(x) \in B$.) Problems that are NP-hard and belong to NP are called NP-*complete*. A polynomial-time algorithm for an NP-hard problem would imply polynomial-time algorithms for all problems NP—something that is considered very unlikely. Therefore, the NP-hardness of a problem is considered substantial evidence for the computational difficulty of the problem.

A standard example of an NP-complete problem is 3SAT, i.e., checking whether a given propositional formula given as a 3CNF has a satisfying assignment. To be more precise, a kCNF is a formula $H = C_1 \wedge \cdots \wedge C_m$ consisting of clauses C_i each of which has the form $C_i = l_{i1} \vee l_{i2} \vee \cdots \vee l_{ik}$ where l_{ij} is either a positive or a negative literal. A positive literal is some variable, say x_k, and a negative literal is the negation of some variable, say $\overline{x_k}$.

The class of complements of NP sets is denoted by coNP, i.e., coNP $= \{\overline{A} | A \in \mathrm{NP}\}$.

For optimization problems (where the goal is to compute a feasible solution that maximizes or minimizes some objective function), we say that the problem is NP-*hard* if the corresponding decision problem (checking whether a solution with objective value better than a given value k exists) is NP-hard.

#P-*completeness*

A complexity class closely related to NP is the class #P which has been introduced in [46, 47] to provide evidence for the computational intractability of counting problems. The class #P consists of all problems of the form "compute $f(x)$" where $f(x)$ is the number of accepting paths of a nondeterministic Turing machine running in polynomial time. Equivalently, a #P-functions counts the number of solutions to instances of an NP-problem. We say that a function f is #P-*complete* if it belongs to #P and every function $g \in$ #P is polynomial-time Turing reducible to f, i.e., g can be computed by a deterministic polynomial-time Turing machines which is allowed to make queries to f and answering these queries is done within one step (see, e.g., [23, 21]). The canonical example of a #P-complete problem is #3SAT, i.e., counting the number of satisfying assignments of a propositional formula given as a 3CNF. One of the most prominent #P-complete problem is counting the number of perfect matchings in a bipartite graph [47]. As in the case of NP, if there is a polynomial-time algorithm for computing some #P-complete function from #P then there are polynomial-time algorithms for all #P-functions—which is equally considered unlikely. In particular, such a polynomial-time algorithm would imply that P = NP.

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