University of Konstanz
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Assignment 10

Post Date: 9 Jan 2013 **Due Date:** 16 Jan 2013, 14:30
You are permitted and encouraged to work in groups of two.

Problem 1: Height of a Fibonacci Heap — 4 Points

Prove or disprove that the height of a tree of a Fibonacci Heap with n vertices is in $\mathcal{O}(\log n)$.

Problem 2: Fibonacci Heaps and MST — 8 Points

Solve the Minimum Spanning Tree Problem in Assignment 5.1 using the Algorithm of Prim with a Fibonacci Heap. Give for each step the necessary heap operations and show how the heap looks like.

Problem 3: Ranks in a Fibonacci Heap — 8 Points

Consider the sequences $(F_k)_{k\in\mathbb{N}_0}$ and $(S_k)_{k\in\mathbb{N}_0}$, recursively defined by

$$F_0 = 0\ , \quad F_1 = 1\ , \quad \text{and} \quad F_{k+2} = F_{k+1} + F_k \quad \text{for all} \quad k \in \mathbb{N}_0,$$

and

$$S_0 = 1\ , \quad S_1 = 2\ , \quad \text{and} \quad S_k = 1 + 1 + \sum_{i=0}^{k-2} S_i \quad \text{for all} \quad k \in \mathbb{N} \setminus \{1\}.$$

$(F_k)_{k\in\mathbb{N}_0}$ is the so-called *Fibonacci sequence*, and for all $k \in \mathbb{N}_0$, S_k is a lower bound for the number of vertices in a subtree of a Fibonacci heap (see the lecture) with root of rank k.

(a) Prove by induction on k that $S_k = F_{k+2}$ for all $k \in \mathbb{N}_0$.

(b) Prove by induction on k that $F_k = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right)$ for all $k \in \mathbb{N}_0$.

(c) Conclude that $F_{k+2} \geq \left(\frac{1+\sqrt{5}}{2}\right)^k$ for all $k \in \mathbb{N}_0$.

(d) Show that the maximum rank of a root of a Fibonacci heap with n vertices is less than $1.5 \cdot \log n$.