University of Konstanz
Department of Computer & Information Science
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Design and Analysis of Algorithms
WS 2012/2013

Assignment 13

Post Date: 30 Jan 2013 **Due Date:** 6 Feb 2013, 14:30
You are permitted and encouraged to work in groups of two.

Problem 1: Max-Flow Min-Cut **10 Extrapoints**

Consider the following formulation of the maximum-flow problem: Given a flow network consisting of

$$\text{a directed graph } D = (V, E),\ s, t \in V, \text{ and capacities } u : E \longrightarrow \mathbb{R}_0^+,$$

add a new "artificial" arc (t, s) with infinite capacity to E and find real values $f(e), e \in E$

$$\text{maximizing } f(t, s)$$

subject to

$$0 \leq f(e) \leq u(e), \quad e \in E$$

and

$$\sum_{(v,w)\in E} f(v, w) = \sum_{(w,v)\in E} f(w, v), \quad v \in V.$$

(a) Show that the above formulation indeed corresponds to the first definition of the maximum-flow problem that you have learned in this course.

(b) Convert the above formulation into a linear program in standard form.

(c) Let $B = (b_{ij})_{i=1,\dots,n;j=1,\dots,m}$ be the incidence matrix of the directed graph D including the artificial arc (t, s) with some arbitrary ordering $v_1, \dots, v_n$ and $e_1, \dots, e_m$ of the vertices and edges, respectively, i.e., let

$$b_{ij} = \begin{cases} 1, & \text{if } v_i \text{ tail of } e_j \\ 0, & \text{if } e_j \text{ not incident to } v_i \\ -1, & \text{if } v_i \text{ head of } e_j \end{cases}$$

Using B, define the matrix A and the vectors b and c such that your linear program corresponds to the short form

$$\text{maximize } c^T x \text{ subject to } Ax \leq b, x \geq 0.$$

(d) Formulate the dual of the maximum-flow linear program.

(e) Interpret the dual of the maximum-flow linear program as a minimum-cut problem.

[please turn over]

Problem 2: 1-Variable Linear Program **6 Extrapoints**

For $a, b, c, x \in \mathbb{R}$ consider the following linear program.

$$\text{maximize} \qquad cx$$

subject to

$$ax \leq b \qquad \text{and} \qquad x \geq 0.$$

(a) State for which values of a, b, and c the primal and the dual linear program, respectively, are infeasible, unbounded or have an optimal solution with finite objective value, respectively. Relate the cases.

(b) Prove or disprove that in general the dual of an unbounded linear program is infeasible.

Problem 3: Complementary Slackness **4 Extrapoints**

Let $\overline{x}$ be a feasible solution to the primal linear program

$$\text{maximize} \qquad \sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, i = 1, \ldots, m \qquad \text{and} \qquad x_j \geq 0, j = 1, \ldots, n$$

and let $\overline{y}$ be a feasible solution to the corresponding dual linear program. Prove that $\overline{x}$ and $\overline{y}$ are optimal if and only if

$$\sum_{i=1}^{m} a_{ij} \overline{y}_i = c_j \text{ or } \overline{x}_j = 0 \quad \text{for } j = 1, \ldots, n$$

and

$$\sum_{j=1}^{n} a_{ij} \overline{x}_j = b_i \text{ or } \overline{y}_i = 0 \quad \text{for } i = 1, \ldots, m.$$