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Network Dynamics
WS 2010/11

Assignment 2

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Problem 1: Congestion Model 10 Points

We define notions in order to describe traffic congestion scenarios: A (non-atomic) *congestion model* is given by a tuple $(F, S, (c_f)_{f\in F})$ such that

(a) F is a finite, non-empty set of roads,

(b) $S \subseteq \mathcal{P}(F)$ is non-empty set of pathways,

(c) $c_f : [0,1] \to \mathbb{R}_{\geq 0}$ is a cost function for each $f \in F$, i.e., if a fraction x of all users choose road f then each user has cost $c_f(x)$.

A mapping $x : S \to [0,1]$ such that $\sum_{s\in S} x(s) = 1$ is a *user distribution*. For a road $f \in F$ and a user distribution x, the *congestion* $\mu_f(x)$ is defined by

$$\mu_f(x) =_{\text{def}} \sum_{\substack{s\in S\\ f\in s}} x(s).$$

If a user chooses pathway $s \in S$ with respect to user distribution x then the user has *user cost* $c_s(x)$ given by

$$c_s(x) =_{\text{def}} \sum_{f\in s} c_f(\mu_f(x)).$$

The *social cost* $C(x)$ of a user distribution x is given by

$$C(x) =_{\text{def}} \sum_{s\in S} c_s(x)\cdot x(s).$$

Suppose your are given the congestion model $(\{a,b\}, \{\{a\},\{b\}\}, (c_a, c_b))$ with cost functions $c_a(x) = 1$ and $c_b(x) = x^r$ for $r > 1$.

(a) A user distribution x is said to be at *Wardrop equilibrium* if $c_s(x) \leq c_{s'}(x)$ for all $s, s' \in S$ such that $x(s) > 0$. Determine a user distribution at Wardrop equilibrium for the above congestion model. What is the social cost of the user distribution?

(b) A user distribution x^* is said to be a *social optimum* if $C(x^*) \leq C(x)$ for all user distributions x. Determine a social optimum. What is the social cost of the social optimum?

(c) Let x^* be a social optimum and let x be a user distribution at Wardrop equilibrium. The *coordination ratio* γ^* is the infimum of all γ such that $C(x) \leq \gamma \cdot C(x^*)$. Determine the coordination ratio for the above congestion model.

Problem 2: Phase Space **10 Points**

Consider the state process $(X, E, R_i)_{i \in [\infty]}$ with population $X = \{0, 1, 2, 3\}$ of type $D = \{0, 1\}$, structure $E = K_{1,3}$ (i.e., the set of edges $\{0,1\}, \{0,2\}, \{0,3\}$), and $R_i = D^4$. Suppose a set $L = \{f_0, f_1, f_2, f_3\}$ of local transitions is given by

$$\begin{aligned} f_0(x_0, x_1, x_2, x_3) &=_{\text{def}} x_0 \oplus x_1 \oplus x_2 \oplus x_3, \\ f_1(x_0, x_1) &=_{\text{def}} x_0 \oplus x_1, \\ f_2(x_0, x_2) &=_{\text{def}} x_0 \oplus x_2, \\ f_3(x_0, x_3) &=_{\text{def}} x_0 \oplus x_3, \end{aligned}$$

where $\oplus$ denotes addition modulo 2.

Describe the phase space of the global state dynamic induced by the local state dynamics (L, π) where $\pi = (1\ 2\ 3\ 0)$.

Problem 3: Update Order **10 Points**

Given a state process $(X, E, R_i)_{i \in [\infty]}$ with population $X = \{0, 1, \ldots, n-1\}$ of an arbitrary type D, structure $E = \mathsf{Circ}_n$, and $R_i = D^n$, determine the maximum number of different global state dynamics induced by local state dynamics (L, π) where L is a set of local transitions over the interdependence structure Circ_n and $\pi : \{0, 1, \ldots, n-1\} \rightarrow \{0, 1, \ldots, n-1\}$ is a permutation.

Hint: Find an appropriate recursive formula and prove your conjecture inductively.