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Network Dynamics
Winter 2015/16

Assignment 4

Ausgabe: 11 Nov 2015 **Abgabe:** 18 Nov 2015

Problem 1: Orbits

Let $A \in \mathbb{R}^{2\times 2}$ be a matrix of the following type:

$$A =_{\text{def}} \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}, \qquad a, b \in \mathbb{R}$$

Consider the function $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2 : x \mapsto A \cdot x$ (which depends on concrete choices of a, b).

Find all pairs (a, b) of values $a, b \in \mathbb{R}$ such that the orbits of $(-1, 1)$ and $(1, 1)$ under $\mathbf{F}$ are disjoint.

Problem 2: Fixed points

We consider the threshold model of collective behavior. Let $A = \{1, \ldots, n\}$ be a population with individual thresholds $t_1 \leq t_2 \leq \cdots \leq t_n$. We define $\mathbf{F} : \{0,1\}^A \to \{0,1\}^A$ to be that mapping that is component-wise given as follows:

$$\mathbf{F}(r)_i =_{\text{def}} \begin{cases} 1 & \text{if } \sum_{j \in A \setminus \{i\}} r_j \geq t_i \\ 0 & \text{otherwise} \end{cases}$$

We may assume that all thresholds lie between 0 and $n-1$. We additionally define $t_0 =_{\text{def}} -1$ and $t_{n+1} =_{\text{def}} n + 1$. We say that there is a *gap at k* if and only if $t_k < k$ and $t_{k+1} \geq k + 1$.

Prove or disprove the following statement:

> Let $r = (r_i)_{i \in A} \in \{0,1\}^A$ satisfy that if $r_i = 1$ and $r_j = 0$ then $i < j$. Let $|r|_1$ denote the number of 1's in r. Then,
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> $$r \text{ is a fixed point of } \mathbf{F} \iff \text{there is a gap at } |r|_1$$

Problem 3: Basins of attraction

Let $\mathbf{F} : J \to J$ be a total map on a finite set J. Consider the following two definitions for some non-empty set $E \subseteq J$:

- E is said to be a *weak invariant set* of $\mathbf{F}$ if and only if $\mathbf{F}(E) \subseteq E$.
- E is said to be a *strong invariant set* of $\mathbf{F}$ if and only if $\mathbf{F}^k(E) \subseteq E$ for all $k \in \mathbb{N}$.

Which type of an invariant set—weak or strong—provides a guarantee for containment of an attractor? Prove your hypothesis.