Fachbereich Informatik und
Informationswissenschaft

Universität
Konstanz

Lecture Notes
Network Dynamics

taught in Winter term 2015

by

Sven Kosub

February 9, 2016

Version v5.14

Contents

An example of social influence: Relative agreement 1

A simple model to study aspects of social influence is the Relative Agreement model of Deffuant, Amblard, Weisbuch, and Faure [2]. The model has been introduced to address the question whether it is possible to identify certain network parameters that endogeneously govern the distribution of opinions within a human population. A particular goal was to look for parameter values that allow extreme opinions to dominate eventually within a human population. The model is representative for a physics-oriented approach to complex networks:

(a) Methodologically, it employs *agent-based modelling*. Agent-based modelling uses simplified interaction models and simulations to explore a nonlinear dynamical behavior of complex systems. Agent-based modelling is applied when kinetic models involving differential equation systems are inappropriate, e.g., due to the number and the heterogeneity of variables.

(b) It explains a complex phenomenon in a *stylized* fashion.

Apart from the methodological perspective, the concrete, original research motivation for the model lies in the influence "green" farmers have attained in the farming population.

1.1 The formal model

We consider the following formal scenario: A population of n agents is given. An agent i is characterized by two variables:

- opinion $x_i \in [-1, 1]$
- uncertainty $u_i > 0$

Thus, the actual opinion of the agent ranges in her opinion segment

$$S_i =_{\text{def}} [x_i - u_i, x_i + u_i],$$

the size of which is $(x_i + u_i) - (x_i - u_i) = 2u_i$.

We suppose a directed model of influence where any two agents use a communication channel. Agent i locally communicates to agent j over her communication channel, possibly causing changes in opinion and uncertainty of agent j. In this situation, agent i is the influencer of agent j and agent j is the influenced of agent i.

The effect of a communicative influence is given by an update rule which is assumed to be the same for all interaction pairs of agents. Figure 1.1 describes a situation of an interaction pair (i, j).

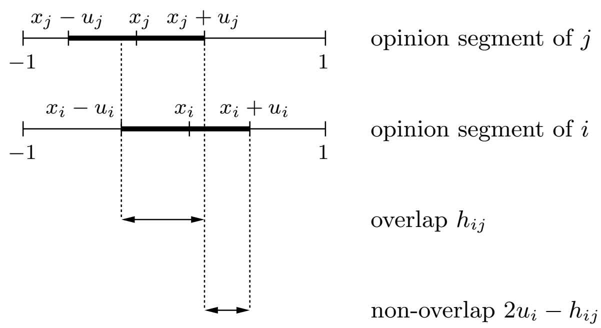


Figure 1.1: The Relative Agreement model

The update rule is based on the agreement along the opinion segments of agents i and j, i.e.,

$$h_{ij} - (2u_i - h_{ij}) = 2(h_{ij} - u_i),$$

in relation to the uncertainty of the influencer

$$\frac{2(h_{ij} - u_i)}{2u_i} = \frac{h_{ij}}{u_i} - 1.$$

The formal specification of the update rule is given by defining local transitions:

$$x_j \leftarrow x_j + \begin{cases} \mu \left(\frac{h_{ij}}{u_i} - 1\right)(x_i - x_j) & \text{if } h_{ij} \geq u_i \\ 0 & \text{otherwise} \end{cases}$$

$$u_j \leftarrow u_j + \begin{cases} \mu \left(\frac{h_{ij}}{u_i} - 1\right)(u_i - u_j) & \text{if } h_{ij} \geq u_i \\ 0 & \text{otherwise} \end{cases}$$

Here, μ is some decay constant, $0 < \mu \leq 1$.

Proposition 1.1 *Let an interaction pair (i, j) be given. Let h_{ij} denote the overlap of the opinion segments of the actors i and j before interaction, and let h'_{ij} denote the overlap of the opinion segments of the actors i and j after interaction. Then, $h_{ij} \leq h'_{ij}$.*

Proof: Let (x_i, u_i) be the opinion/uncertainty pair of actor i, let (x_j, u_j) be the opinion/uncertainty pair of actor j. According to our update rule, if $h_{ij} \leq u_i$ then there are no changes, neither in the opinions nor in the uncertainties of both actors. That is, $h'_{ij} = h_{ij}$.

So, let $h_{ij} > u_i$. Let x'_j denote actor j's opinion after interaction, and let u'_j denote actor j's uncertainty after interaction. More specifically, we have

$$\begin{aligned} x'_j &= \left(1 - \mu\left(\frac{h_{ij}}{u_i} - 1\right)\right) x_j + \mu\left(\frac{h_{ij}}{u_i} - 1\right) x_i, \\ u'_j &= \left(1 - \mu\left(\frac{h_{ij}}{u_i} - 1\right)\right) u_j + \mu\left(\frac{h_{ij}}{u_i} - 1\right) u_i. \end{aligned}$$

The overlap h'_{ij} is given by

$$h'_{ij} = \min(x_i + u_i, x'_j + u'_j) - \max(x_i - u_i, x'_j - u'_j).$$

Note that $h_{ij} \le 2u_i$. Thus, the update rules define convex combinations. By linearity, we can easily examine four cases depending on the locations of the boundaries of the opinion segments of both agents (Exercise). This proves the proposition. ■

Proposition 1.2 *Let the interaction pair (i, j) be given. For $k \in \mathbb{N}$, let $x_j^{(k)}$ be actor j's opinion after the k-th round of the directed interaction (i, j), and let $u_j^{(k)}$ be actor j's uncertainty after the k-th round of the directed interaction (i, j). Then,*

$$\lim_{k\to\infty} x_j^{(k)} = x_i, \qquad \lim_{k\to\infty} u_j^{(k)} = u_i.$$

Proof: We only prove the convergence in opinions. Without loss of generality, we may assume that $x_i \ge x_j$. Since $h_{ij} \le 2u_i$, we obtain as an upper bound on the opinion $x_j^{(k)}$ for $k \in \mathbb{N}_+$

$$x_j^{(k)} \le (1-\mu)x_j^{(k-1)} + \mu x_i,$$

and furthermore, by induction,

$$x_j^{(k)} \le (1-\mu)^k x_j + \left(1 - (1-\mu)^k\right) x_i.$$

Hence, $\lim_{k\to\infty} x_j^{(k)} \le x_i$. For the lower bound, we write

$$x_j^{(k)} = (1 - \mu A_{k-1})x_j^{(k-1)} + \mu A_{k-1} x_i,$$

where $A_k = \frac{h_{ij}^{(k)}}{u_i} - 1$ is the relative agreement after the k-th interaction. By Proposition 1.1, it holds that $A_k \le A_{k+1}$ for all $k \in \mathbb{N}$. Thus, we can estimate

$$x_j^{(k)} \ge (1 - \mu A_0)x_j^{(k-1)} + \mu A_0 x_i,$$

and, again by induction,

$$x_j^{(k)} \ge (1 - \mu A_0)^k x_j + \left(1 - (1 - \mu A_0)^k\right) x_i,$$

Hence, $\lim_{k\to\infty} x_j^{(k)} \ge x_i$. This proves the proposition. ■

1.2 The role of extremists

In general populations of actors, it is not clear at all whether there is any convergence to a "stable" opinion/uncertainty pattern over several time steps. If unambiguous convergence is reachable, there are three important cases: convergence to the opinion poles, either positive or negative, or convergence to the middle. We are interest in studying convergence to extreme opinions.

Extremists are people with extreme opinions, i.e., opinions close to the boundaries measured by -1 and 1. Furthermore, the model of extremists within a population is based on two observations which are claimed to possess a certain anectodal evidence [2]:

1. "... people who have extreme opinions tend to be more convinced,"
2. "... people who have moderate initial opinions, often express a lack of knowledge (and uncertainty)."

A simplification of these observations can be incorporated into the Relative Agreement model as follows. Let u_e be the uncertainties of the extremists, supposed to be small and the same for all extremists. Let u be the (identical) uncertainty of the moderate. According to our observations, it holds that $u > u_e$. Then, the population can be initially decomposed into three classes corresponding to their opinion/uncertainty pairs:

1. *positive extremists*: $x_i \approx 1$, $u_i = u_e$
2. *negative extremists*: $x_i \approx -1$, $u_i = u_e$
3. *moderates*: $x_i \approx 0$, $u_i = u$

Let p_e denote the fraction of extremists, either positive or negative, in the population. Depending on the fraction of actors belonging to these classes, an extremism bias can be defined. Let p_+ be the fraction of positive extremists, and let p_- be the fraction of negative extremists. Then, the *extremism bias* δ is given as

$$\delta = \frac{p_+ - p_-}{p_+ + p_-}$$

The simulation works in two phases:

1. For initalization, (a) choose n opinions uniformly at random from $[-1, 1]$ and set all n uncertainties to u, (b) for the $p_+ \cdot n$ most positive opinions and $p_- \cdot n$ most negative opinons, the uncertainties are set to u_e.
2. Iteratively choose a pair (i, j) of agents and let agent i exert influence on agent j according to the specified update rule.

The stylized simulation results can be divided into three stable scenarios: central clustering, bipolarization, single polarization. The following figures show diagram schemes for each of the three scenarios together with parameter settings such that the described behavior can be observed. The x-axis codes for time, i.e., number of iterations per actor, and the y-axis codes for opinions. A trajectory of an actor's opinion over the course of time runs inside the region bounded by the drawn curves. Common parameters for all figures (and the simulations) are $n = 200$, $\mu = 0.5$, $\delta = 0$, and $u_e = 0.1$. The initial uncertainty parameter u is increased from figure to figure.

In Figure 1.2, the initial uncertainty of the moderates is $u = 0.4$. It is an example of central convergence. The majority of the moderate actors are not attracted by the extreme opinions.

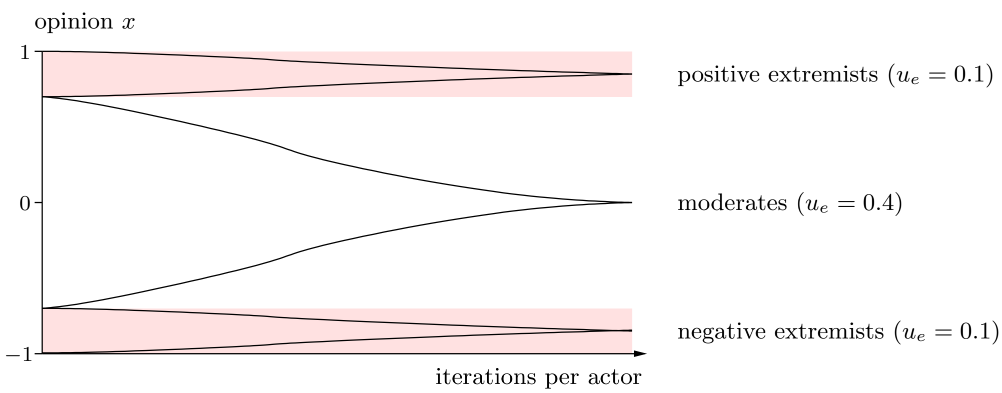


Figure 1.2: Scheme of central convergence.

In Figure 1.3, the initial uncertainty of the moderates is $u = 1.2$. It is an example of convergence to both extremes. The initially moderate actors split and become extremists.

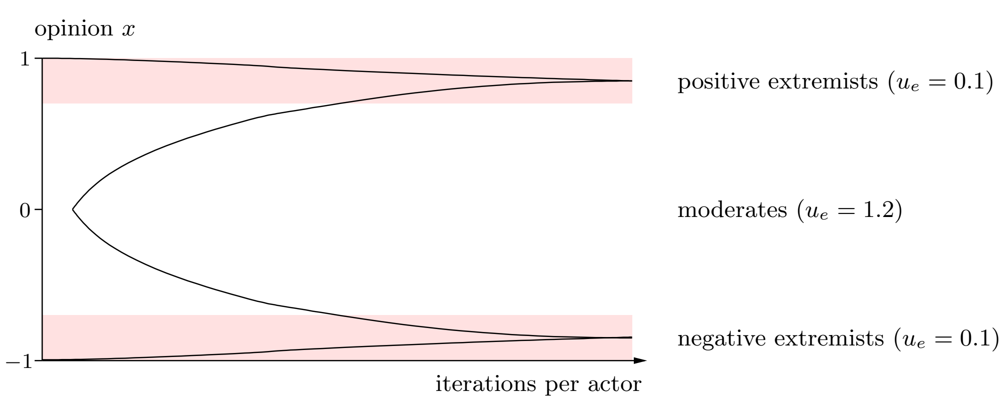


Figure 1.3: Scheme of bipolarization.

In Figure 1.4, the initial uncertainty of the moderates is $u = 1.4$. It is an example of convergence to one single extreme. In this case, the majority of the population is attracted by one of the extremes. This behavior can take place even when the number of initial extremissts is the same at both extremes, which is claimed to have been a priori unexpected.

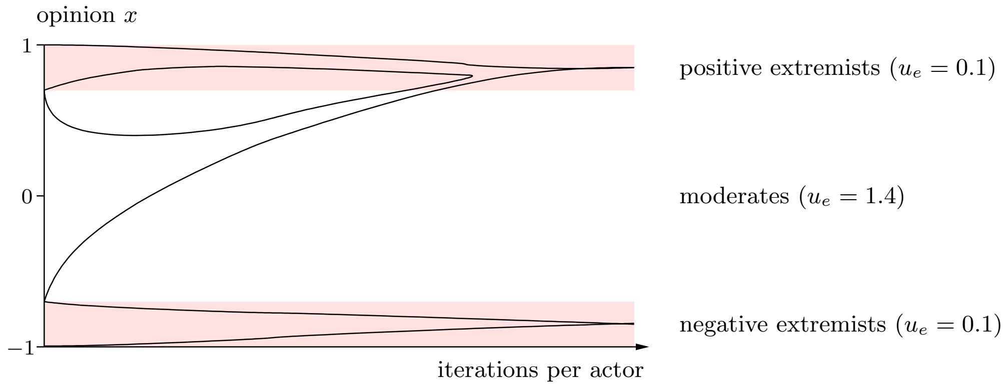


Figure 1.4: Scheme of single polarization.

Networks as a dynamical systems 2

2.1 Networks from data

2.1.1 Network data

Data. *Data* refers to variables for *entites* (or *units of observation*). More specifically,

- A is a set of (atomic) *items*,
- for $i \in A$, variable x_i represents values of a common *attribute* for all items in A, i.e., x is a mapping $x : A \to R : i \mapsto x_i$, or $x = (x_i)_{i \in A}$ where $x_i \in R$,
- R is the *range* of x, A is called the *domain* of x

Typically in emprical research, multiple attributes are collected in tables where the columns represent items and the rows represent attributes.

> **Example:** Consider a set $A = \{\text{Abner}, \text{Ashkan}, \text{Betty}, \text{Elshad}, \text{Evette}, \text{Kenzo}\}$ of six students. Suppose we are interested in the working days each student is regularly present at the university during a week in a certain semester. This data is given by the following table:
>
> | A | mon | tue | wed | thu | fri |
> |---:|---|---|---|---|---|
> | Abner | x | | x | x | x |
> | Ashkan | | x | x | x | |
> | Betty | x | | x | | x |
> | Elshad | x | | x | | |
> | Evette | | x | | x | x |
> | Kenzo | | x | | x | |
>
> Clearly, we have five attributes with range $\{x, \ \}$ defined on A.

According to the range, attributes can be classified:

- *nominal* or *categorical*: there are no relationships among the elements of the range other than equality or inequality (e.g., names, types, labels)
- *ordinal*: the range satisfy certain order properties such as required for weak orders, preference relations, rankings (e.g., paths in policy routing)

- *numerical*: the range consists of number such as $\mathbb{N}$ or $\mathbb{R}_{\geq 0}$.

We assume that 0 represents a missing or neutral datum.

Dyadic data. Entities need not be atomic; they can be compound objects of more elementary entities. More specifically,

- a *dyad* is a pair of items,
- two dyads *overlap* if and only if they share a member,
- *network data* is the dyadic data characterized by
  1. units of observation are dyads and
  2. dyads are overlapping.

As a remark, dyadic data analysis assumes independence of dyads (in a statistical sense). In contrast, network analysis is fundamentally based on dependence of dyads. In this regard network data are a more general concept than dyadic data. For instance, in a study we could explore relationship among married couples. The relevant data may include: attributes of the individual, e.g., gender, income, personality; attributes of the couple, e.g., age difference, duration of marriage, number of children; or, attributes of the pair of individuals,e.g., time spent together, number of mutual friends. The first class of attributes refers to (atomic) data, the second class of attributes refers to dyadic data, and the third class of attributes refers to network data.

> **Example:** We continue our student example. Given the five (atomic) attributes mon, tue, wed, thu, fri: $A \to \{x, \quad\}$, we are interested in the number of days, two students can meet at the university. This is network data.

Time-dependent data. Attribute values may change over time. And, there are differences in how data can depend on time. In general, time-dependent data can be classified as follows:

- *panel data* (or *longitudinal data*): we have attributes values of all items for at least two points in time, i.e., $x(1), x(2), \ldots, x(k)$ where $x(j) = (x_i(j))_{i \in A}$.
- *time-series data*: we have attribute values of a single item over time.
- *cross-sectional data*: we have attribute values of all items for one specific point in time.
- *event data*: we have attribute values for items labelled with a time stamp (e.g., log files, audit trails, live scores, etc.)

Typically, event data are transformed into panel data.

2.1.2 Network representations

We adopt a network view where we consider networks to be representations of a specific format. That is, we are not so much interested in *what* is represented, but *how* it is represented. As overlapping dyads are the fundamental objects of network analysis, we need a notion to collect all possible dependencies among dyads. This is done by introducing interaction domains.

Definition 2.1 *Let A be a set of items. An* interaction domain $\mathcal{I}$ *on A is a binary, symmetric relation $\mathcal{I} \subseteq A \times A$.*

In many cases, $\mathcal{I} = A \times A$ or $\mathcal{I} = (A \times A) \setminus \{\ (i,i) \mid i \in A\ \}$.

Definition 2.2 *Let A be a set of items. A* (whole) network *consists of a set of attributes on an interaction domain $\mathcal{I} \subseteq A \times A$ and a (possibly empty) set of attributes on A.*

For a network, items of A represent *actors*, and attribute values $x_{i,j} \neq 0$, where $(i,j) \in \mathcal{I}$, are *ties*. Notice that x_{ij} is a usual abbreviation for $x_{(i,j)}$ for any dyad $(i,j) \in \mathcal{I}$. Attributes on $\mathcal{I}$ are called *network attributes*; attributes on A are called *behavioral attributes*.

> **Example (cont'd):** In our student example, the attributes mon, tue, wed, thu, and fri are behavioral attributes. We are interested in studying the co-presence network x given by the number of days two students visit the university. The interaction domain is $\mathcal{I} = (A \times A) \setminus \{\ (i,i) \mid i \in A\ \}$. Then, the network attribute $x : \mathcal{I} \to \mathbb{N}$ is defined for all dyads $(i,j) \in \mathcal{I}$ as
>
> $$x_{ij} =_{\text{def}} \|\{\ k \mid \text{day}_k(i) = \text{day}_k(j) = \text{x}\ \}\|,$$
>
> where $\text{day}_1 = \text{mon}$, $\text{day}_2 = \text{tue}$, $\text{day}_3 = \text{wed}$, $\text{day}_4 = \text{thu}$, and $\text{day}_5 = \text{fri}$.

There are three standard representations of networks. In the following, we discuss the graph, matrix, and relational representation for a single network attribute. Let $x : \mathcal{I} \to R$ be a network attribute defined on an interaction domain $\mathcal{I} \subseteq A \times A$.

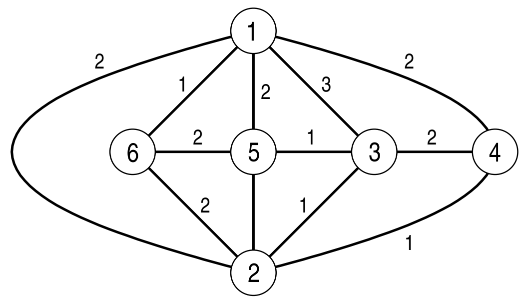
Graphs. The (weighted, directed) *graph $G(x) = (V, E, w)$ of network x* consists of

- vertex set $V =_{\text{def}} A$,
- edge set $E =_{\text{def}} \{\ (i,j) \in \mathcal{I} \mid x_{ij} \neq 0\ \}$, and
- edge weights $w : E \to R : (i,j) \mapsto x_{ij}$.

If $x_{ij} = x_{ji}$ for all $(i, j) \in \mathcal{I}$, $G(x)$ can be defined correspondingly as an undirected graph.

Example (cont'd): Let us assume that the set A of students is enumerated $1, \ldots, 6$. The columns are enumerated $1, \ldots, 5$, corresponding to the day in the week. Furthermore, we transform our initial data into binary data (on the left-hand side). Then, the network attribute x can be represented by the undirected graph on the righ-hand side.

A	1	2	3	4	5
1	1	0	1	1	1
2	0	1	1	1	0
3	1	0	1	0	1
4	1	0	1	0	0
5	0	1	0	1	1
6	0	1	0	1	0



Matrices. A completion of an attribute to the full interaction domain $A \times A$ by imputing zeroes gives the *adjacency matrix* of the associated weighted graph, which is another representation of a network.

Example (cont'd): The network attribute x can be represented by the weighted adjacency matrix on the left and the unweighted adjacency matrix on the right:

$$\begin{pmatrix} 0 & 2 & 3 & 2 & 2 & 1 \\ 2 & 0 & 1 & 1 & 0 & 2 \\ 3 & 1 & 0 & 2 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 2 \\ 1 & 2 & 0 & 0 & 2 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Note that, by abuse of notation, we denote both matrices with $A(x)$; the specific meaning is obtained from the context. Note also that there is a difference between the zeroes in the matrix. The zeroes along the diagonal are artifacts, whereas the zeroes beyond the diagonal are values of the network attribute.

Relations. The (binary) *relation* $\rightarrow \subseteq A \times A$ *of network* x is defined by

$$(i, j) \in \rightarrow \quad \Longleftrightarrow_{\text{def}} \; (i, j) \in \mathcal{I} \wedge x_{ij} \neq 0$$

In infix notation, this is written as $i \rightarrow j$.

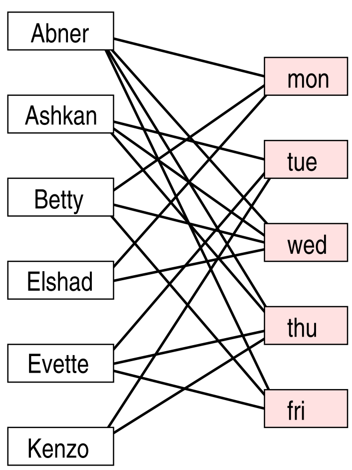
In the following we extend these notions and representations to another type of networks: *two-mode networks*. Assume that the observational units are relations between pairs of items of different types, e.g., users and fan sites on Facebook, authors and scientific papers, or politicians and boards. We generalize interaction domains.

Definition 2.3 *An* affiliation domain *is a relation $\mathcal{A} \subseteq A \times S$ on disjoint sets A and S.*

Definition 2.4 *A* two-mode network *consists of a set of attributes on an affiliation domain $\mathcal{A} \subseteq A \times S$ and a (possibly empty) set of attributes on A and S.*

All notions for networks translate to two-mode networks. Note that two-mode networks are bipartite by definition.

Example (cont'd): In our student example, the initial data can be modelled as a two-mode network. The set A remains the set of students identified either by name or by number. Define $S =_{\text{def}} \{\text{mon}, \text{tue}, \text{wed}, \text{thu}, \text{fri}\}$. The table with the initial data corresponds to the incidence matrix of a two-mode network $x : A \times S \to \{0, 1\}$ represented by the following bipartite graphs:



Definition 2.5 *Let $X \in R^{n \times m}$ be the matrix associated with a two-mode network attribute, $\|A\| = n$ and $\|S\| = m$. The networks associated with the matrices $X \cdot X^T$ and $X^T \cdot X$ are called* one-mode projections.

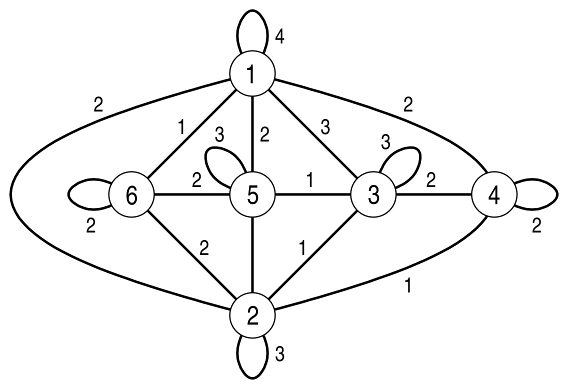
Note that the interaction domain of $X \cdot X^T$ is $A \times A$ and the interaction domain of $X^T \cdot X$ is $S \times S$.

Example: Consider the sets $A = \{1, 2, 3, 4, 5, 6\}$ and $S = \{1, 2, 3, 4, 5\}$, i.e., A represents the students, S represents the week days. Let X denote the

6×5 incidence matrix given by the binary initial data. Then, we obtain the following one-mode projection on the side of students:

$$X \cdot X^T = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 2 & 3 & 2 & 2 & 1 \\ 2 & 3 & 1 & 1 & 2 & 2 \\ 3 & 1 & 3 & 2 & 1 & 0 \\ 2 & 1 & 2 & 2 & 0 & 0 \\ 2 & 2 & 1 & 0 & 3 & 2 \\ 1 & 2 & 0 & 0 & 2 & 2 \end{pmatrix}$$

The weighted graph of the one-mode projection looks like follows:



Notice that the (weighted) loops are the only difference to the originally defined network attribute x.

The one-mode projection on the side of the days is given by:

$$X^T \cdot X = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 3 & 1 & 2 \\ 0 & 3 & 1 & 3 & 1 \\ 3 & 1 & 4 & 2 & 2 \\ 1 & 3 & 2 & 4 & 2 \\ 2 & 1 & 2 & 2 & 3 \end{pmatrix}$$

It contains information on how many of the students visit the university on one (along the diagonal) or two (beyond the diagonal) specific days. For instance, zero indicates that no student can be seen on both monday and tuesday at the university.

2.1.3 Time-dependent networks

We consider attributes on an interaction (or affiliation) domain changing over time. The focus is on panel network-data.

Definition 2.6 *A* time-dependent network *is a set of attributes on an interaction domain $\mathcal{I} \subseteq A \times A$ and a (possibly empty) set of attributes on A, where all atributes depend on (same) time $t \in \mathbb{N}$.*

Note that we consider time-discrete networks.

In order to study time-dependent behavior of networks, we introduce specific formal notions. We restrict ourselves to single-attribute networks with a fixed interaction domain but will later discuss more complex examples.

Let $x : \mathcal{I} \to R$ be a network attribute. For the sake of convenience, we assume that x is a numerical attribute. Furthermore, we consider an infinite sequence of identical copies of x, i.e., $(x(t))_{t\in\mathbb{N}}$ or $x : \mathcal{I} \times \mathbb{N} \to R$. The attribute values are called *states*. The set of all possible sequences is called a *process*; one specific sequence is called *trajectory*. A *dynamic* F is a mechanism for selecting trajectories of a process. A dynamic makes assumptions on how the state at time step k will look like; here, depending only on the initial state z_0 and time k. We thus can express a dynamic as a sequence $(\varphi_t)_{t\in\mathbb{N}}$ where $\varphi_k : R^I \to R^I$.

We adopt notions and notations from dynamical systems. That is, the functions φ_k are iterated maps. Let $F : R^I \to R^I$ be any function. Then, inductively define

$$F^0(z) =_{\text{def}} z, \qquad F^k(z) =_{\text{def}} F(F^{k-1}(z)) \text{ for } k > 0.$$

So, $\varphi_k = F^k$.

The following summarizes the notions schematically:

$$\begin{array}{ccccccccccc}
x(0) & \to & x(1) & \to & x(2) & \to & \dots & \to & x(k) & \to & \dots & \color{red}{\text{process}} \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow & & & \\
z_0 & \to & z_1 & \to & z_2 & \to & \dots & \to & z_k & \to & \dots & \color{red}{\text{trajectory}} \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow & & & \\
\varphi_0(z_0) & \to & \varphi_1(z_0) & \to & \varphi_2(z_0) & \to & \dots & \to & \varphi_k(z_0) & \to & \dots & \color{red}{\text{dynamic}} \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow & & & \\
F^0(z_0) & \to & F^1(z_0) & \to & F^2(z_0) & \to & \dots & \to & F^k(z_0) & \to & \dots & \color{red}{\text{iterated map}}
\end{array}$$

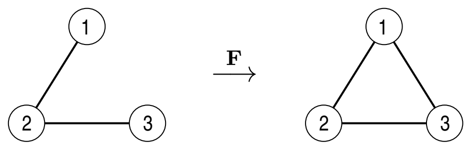
Notice that iterated maps describe memory-less dynamics.

2.2 Iterated network maps

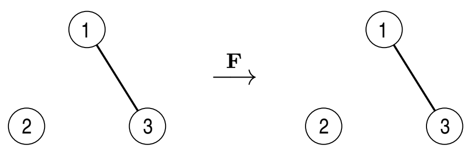
We study dynamics induced by iterating a map $\mathbf{F} : R^{\mathcal{I}} \to R^{\mathcal{I}}$, i.e., a network x is mapped to a network $\mathbf{F}(x)$. In general, we focus on network attributes but, as networks are collections of attribute, examples of behavioral attributes and examples of networks with more than one attribute are discussed as well.

> **Example:** (*Link prediction*) In link prediction, given a snapshot of a social network, we are interested in what new interactions are likely to occur in near future [14]. A mechanism often assumed is the tendency of social networks to triadic closure, i.e., closing open triangles. Suppose we are given the (symmetrical) interaction domain $\mathcal{I} = \{\ \{i,j\} \mid i,j \in A, i \neq j\ \}$ for a set

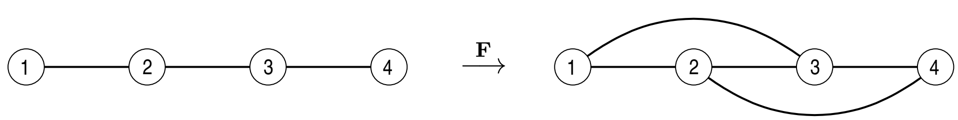
$A = \{1, \ldots, n\}$. An *open triangle* of the (undirected) network $x : \mathcal{I} \to \{0, 1\}$ is formed by three dyads $\{i, j\}$, $\{j, k\}$, and $\{i, k\}$, where i, j, k are pairwise different, such that $x_{\{i,j\}} = x_{\{j,k\}} = 1$ and $x_{\{i,k\}} = 0$. In the graph of network, an open triangle is thus an induced path P_3, i.e., a path consisting of three nodes and two edges. Define $\mathbf{F} : \{0, 1\}^{\mathcal{I}} \to \{0, 1\}^{\mathcal{I}}$ to be the map that closes all open triangles of a given network x, i.e., if dyads $\{i, j\}$, $\{j, k\}$, and $\{i, k\}$ form an open triangle in the network x then in network $F(x)$, it holds that $\mathbf{F}(x)_{\{i,j\}} = \mathbf{F}(x)_{\{j,k\}} = \mathbf{F}(x)_{\{i,k\}} = 1$. For instance, an open triangle is closed in the first network



whereas there is no open triangle in the second network:



Of course, there can be more than one open triangle in a network:



Changes of behavioral attributes are often referred to as dynamics *on* networks, in contrast to dynamics *of* networks which describes the change of network attributes.

Example: (*Riots*) The example was given by Granovetter [8]. We consider the following behavioral attribute: Let $A = \{1, \ldots, n\}$ be any set of persons, $\mathcal{I} = (A \times A) \setminus \{\ (i, i) \mid i \in A\ \}$. Each person i has a certain threshold t_i, i.e., if t_i persons in her neighborhood have joined the riot, person i joins as well. So, $r : A \to \{0, 1\}$ is the attribute indicating participation in the riot for each person. Define $\mathbf{F} : \{0, 1\}^A \to \{0, 1\}^A$ to be the mapping that is component-wise given as follows:

$$F(r)_i =_{\text{def}} \begin{cases} 1 & \text{if } \displaystyle\sum_{j \in A \setminus \{i\}} r_j \geq t_i \\ 0 & \text{otherwise} \end{cases}$$

Generally, attributes can interact in a complex manner within networks. In the following, we discuss a simple network example with more than one attribute.

Example: (*Job rotation*) In a company, n trainee employees can be assigned to n positions in k departments $D_1, \dots, D_k$. It is supposed that department D_i has n_i positions, so that $n_1 + \cdots + n_k = n$. Assume that positions are enumerated in ascending order of the departments, i.e., $D_i =_{\text{def}} \{N_{i-1}+1, \dots, N_i\}$ for all $i \in \{1, \dots, k\}$, where $N_i =_{\text{def}} \sum_{j=1}^{i} n_j$ for $i > 0$ and $N_0 =_{\text{def}} 0$. Also assume that trainee employees are enumerated $1, \dots, n$.

We consider two different networks: Let the set A of items consist of trainees, i.e., $A = \{1, \dots, n\}$. An assigment is a bijective mapping $\pi : A \to \{1, \dots, n\}$, i.e., trainee i is currently at position $\pi(i)$. Let the set S of affiliations consist of departments, i.e., $S = \{D_1, \dots, D_k\}$.

- A two-mode network $x : A \times S \to \{0, 1\}$ is defined as follows:

$$x(i, D_j) = 1 \iff_{\text{def}} \pi(i) \in D_j$$

- A one-mode network $y : A \times A \setminus \{\ (i, i) \mid i \in A\} \to \{0, 1\}$ is defined as follows:

$$y(i, j) = 1 \iff_{\text{def}} \text{there is an } \ell \in \{1, \dots, k\} \text{ such that } \{\pi(i), \pi(j)\} \subseteq D_\ell$$

We assume that the company rotates trainees cyclically to the next position.

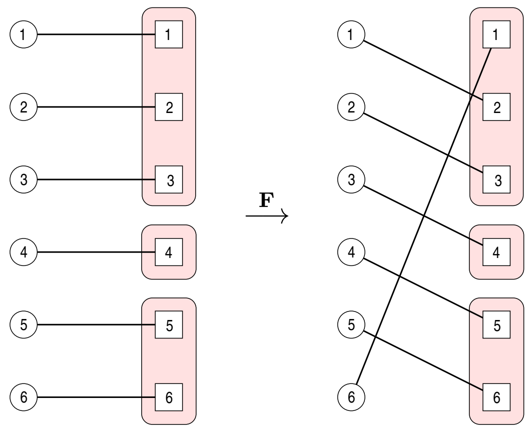
Formally, we consider network $\{x, \pi\}$ and $\{y, \pi\}$. Rotations are maps

$$\begin{aligned} \mathbf{F}_2 &= (F_2, F) &: \ & \{0,1\}^{A \times S} \times \{1, \dots, n\}^A \to \{0,1\}^{A \times S} \times \{1, \dots, n\}^A \\ \mathbf{F}_1 &= (F_1, F) &: \ & \{0,1\}^{\mathcal{I}} \quad\ \times \{1, \dots, n\}^A \to \{0,1\}^{\mathcal{I}} \quad\ \times \{1, \dots, n\}^A \end{aligned}$$

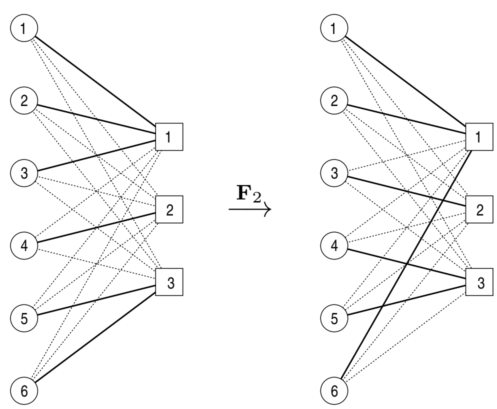
which can be component-wise defined as follows:

$$\begin{aligned} F(\pi)_i \quad &=_{\text{def}} \quad 1 + \big(\pi(i) \bmod n\big) \\ F_2(x)_{ij} = 1 \quad &\iff_{\text{def}} \quad F(\pi)_i \in D_j \\ F_1(y)_{ij} = 1 \quad &\iff_{\text{def}} \quad \text{there is an } \ell \in \{1, \dots, k\} \text{ such that } \{F(\pi)_i, F(\pi)_j\} \subseteq D_\ell \end{aligned}$$

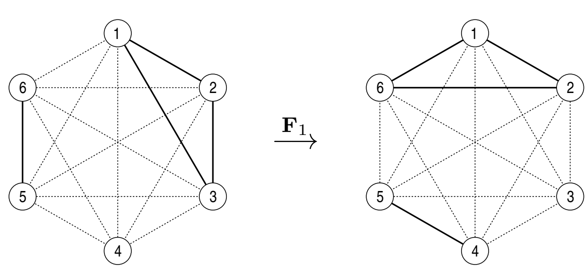
For instance, let $A = \{1, \dots, 6\}$, $D_1 = \{1, 2, 3\}$, $D_2 = \{4\}$, and $D_3 = \{5, 6\}$. The effect of $\mathbf{F}$ on the identity assigment can be depicted as follows:



Given the identity assignment, the corresponding two-mode network x is shown on the left hand-side, and the result of applying the component map $\mathbf{F}_2$ is shown on the right-hand side:



Likewise, the effect to applying the component map $\mathbf{F}_1$ on the one-mode network y can be shown as follows:



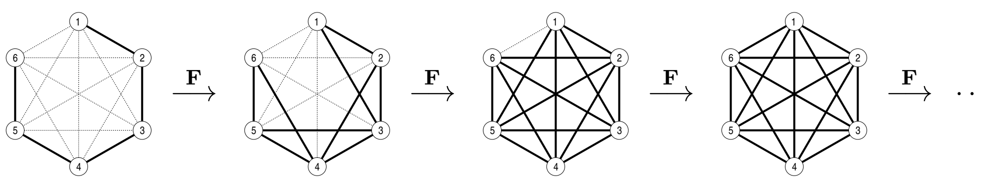
2.2.1 Orbits

A fundamental concept in the study if iterated maps is the orbit.

Definition 2.7 *Let* $\mathbf{F} : J \to J$ *be a total map,* $z_0 \in J$*. The* orbit *of* z_0 *under* $\mathbf{F}$ *is defined to be the sequence* $(z_0, z_1, \ldots, z_k, \ldots)$ *such that* $z_k = \mathbf{F}^k(z_0)$ *for all* $k \in \mathbb{N}$*.*

Note that an orbit is a specific trajectory.

Example: (*Link prediction*) Let $\mathbf{F} : \{0,1\}^{\mathcal{I}} \to \{0,1\}^{\mathcal{I}}$ be as defined. The initial graphs in orbit of some given network $x^{(0)}$ under $\mathbf{F}$ are given by the following sequence (together with the underlying interaction domain):



As we quickly reach a complete graph, there are no more open triangles and, thus, a further application of $\mathbf{F}$ does not change the graph anymore, i.e., $\mathbf{F}^3(x^{(0)}) = \mathbf{F}^4(x^{(0)}) = \cdots = \mathbf{F}^k(x^{(0)})$ for $k \geq 3$.

Example: (*Riots*) Given a small population $A = \{1,2,3,4\}$ with thresholds $t_i = i - 1$ for $i \in A$, the orbit of $r^{(0)} = (0,0,0,0)$ under $\mathbf{F} : \{0,1\}^A \to \{0,1\}^A$ is as follows:

$$(0,0,0,0) \xrightarrow{\mathbf{F}} (1,0,0,0) \xrightarrow{\mathbf{F}} (1,1,0,0) \xrightarrow{\mathbf{F}} (1,1,1,0) \xrightarrow{\mathbf{F}} (1,1,1,1) \xrightarrow{\mathbf{F}} \ldots$$

Again, when iterating $\mathbf{F}$ four times for an initially peaceful population, we have reached a state which does not change any more. That, a riot has formed involving the full population. A possibly (local) media reaction to such a cascading effect might lead to the following headline: "A crowd of radicals engaged in riotous behavior."

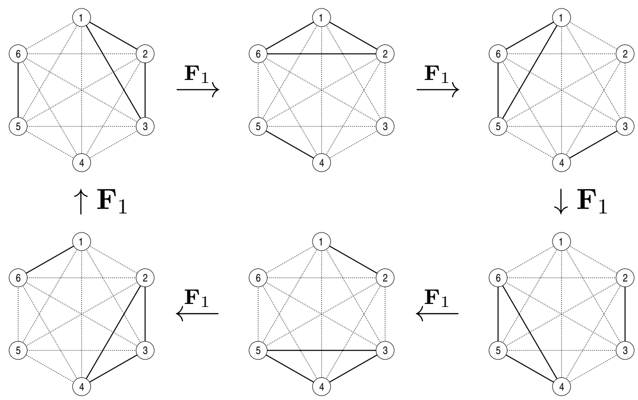
Now, suppose person 2 has a slightly different threshold $t_2 = 2$ while all other thresholds are the same as before. Then, the orbit of $r^{(0)}$ is as follows:

$$(0,0,0,0) \xrightarrow{\mathbf{F}} (1,0,0,0) \xrightarrow{\mathbf{F}} (1,0,0,0) \xrightarrow{\mathbf{F}} \ldots$$

Here, after one iteration the cascading effect stops and we have reached a configuration where no more persons join the riot. A possibly (local) media reaction likely leads to a less lurid headline: "A demented troublemaker broke a window while a group of solid citizens looked on."

As Granovetter [8] pointed out by this example, it is hazardeous to infer individual dispositions from aggregate outcomes.

Example: (*Job rotation*) We consider the orbit of the one-mode network under $\mathbf{F}_1$ as described:



Thus, the concrete values of the network attributes cycle.

Proposition 2.8 *Let $\mathbf{F} : J \to J$ be a total map, and let $x, y \in J$. Then, the orbits of x and y under F are either disjoint or there exist $k \in \mathbb{N}$ and $r \in \mathbb{Z}$ such that $\mathbf{F}^{k'}(x) = \mathbf{F}^{k'+r}(y)$ for all $k' \geq k$.*

Proof: Suppose the orbits of x and y are not disjoint, i.e., there are $t, t' \in \mathbb{N}$ such that $\mathbf{F}^{t}(x) = \mathbf{F}^{t'}(y)$. Define $r =_{\text{def}} t' - t$. So, $t' = t + r$. Then, by induction on $\ell \in \mathbb{N}$, we easily obtain that $\mathbf{F}^{t+\ell}(x) = \mathbf{F}^{t+r+\ell}(y)$ for all $\ell \in \mathbb{N}$. Hence, setting $k =_{\text{def}} t$ and $k' =_{\text{def}} t + \ell$ proves the proposition. ■

Given some map $\mathbf{F} : J \to J$, all orbits under $\mathbf{F}$ are collected in the *phase space*. The fundamental problem (in statistical mechanics) is getting knowledge on the probability distribution over the phase space, i.e., to determine the visiting probablity of a certain state (i.e., an element of J) in an orbit. The following concepts are essential for addressing this question.

Definition 2.9 *Let $\mathbf{F} : J \to J$ be a total map.*

1. *A state $x \in J$ is called* fixed point *of $\mathbf{F}$ if and only if $\mathbf{F}(x) = x$.*
2. *A state $x \in J$ is called* periodic *under $\mathbf{F}$ if and only if there exists a $k \in \mathbb{N}_+$ such that $\mathbf{F}^k(x) = x$. The number $k_0 \in \mathbb{N}_+$ minimal subject to $\mathbf{F}^{k_0}(x) = x$ is called the* periodic order *of x, and x is then called* periodic of order k_0.
3. *A state $x \in J$ is called* transient *under $\mathbf{F}$ if and only if $\mathbf{F}^k(x) \neq x$ or all $k \in \mathbb{N}_+$, i.e., x is not periodic.*

Obviously, a fixed point is periodic of order one.

Example: We discuss our examples in the light of Definition 2.9.

- (*Link prediction*) Each collection of complete graphs is a fixed point under the triadic-closure map $\mathbf{F}$. All other graphs are transient.

- (*Riots*) The number and shape of fixed points depend on individual thresholds. Let $A = \{1, \ldots, n\}$ be a population with thresholds $t_1 \leq t_2 \leq \cdots \leq t_n$. Note that we may assume that all thresholds lie between 0 and n. We define $t_0 =_{\text{def}} -1$ and $t_{n+1} =_{\text{def}} n+1$. We say that there is a *gap at* k if and only if $t_k < k$ and $t_{k+1} \geq k+1$. Then, $r = (r_i)_{i \in A}$ is a fixed point if and only if there is a gap at $|r|_1$ (i.e., the number of 1's in r). All other states are transient.

- (Job rotation) In each network $\{x, \pi\}$ and $\{y, \pi\}$, all states are periodic of order n. Note that this depends on the assignments π which are part of the network.

The following proposition explains why recurring states are referred to as "periodic."

Proposition 2.10 *Let $\mathbf{F} : J \to J$ be a total function. Let $x \in J$ be periodic of order k_0, and let $k \in \mathbb{N}$. Then, the following holds:*

$$\mathbf{F}^k(x) = x \Longleftrightarrow k_0 \textit{ divides } k$$

Proof: We prove both directions individually.

($\Leftarrow$) Observe that $x = \mathbf{F}^{k_0}(x) = \mathbf{F}^{k_0}(F^{k_0}(x))$. An easy inductive argument shows that $x = \mathbf{F}^{c \cdot k_0}(x)$ for all $c \in \mathbb{N}$. Hence, if k_0 divides k, i.e., $k = c \cdot k_0$ for some $c \in \mathbb{N}$, then $\mathbf{F}^k(x) = x$.

($\Rightarrow$) Case $k = 0$ is trivial. Now, suppose $k \geq k_0 > 0$. Then, $k = c \cdot k_0 + r$ for uniquely determined $c \in \mathbb{N}_+$ and $r \in \{0, 1, \ldots, k_0 - 1\}$. Thus,

$$x = \mathbf{F}^k(x) = \mathbf{F}^{c \cdot k_0 + r}(x) = \mathbf{F}^r(\mathbf{F}^{c \cdot k_0}(x)) = \mathbf{F}^r(x)$$

Since k_0 is the smallest positive number with this property, it follows that $r = 0$. Hence, $k = c \cdot k_0$. So, k_0 divides k.

This proves the proposition. ■

Definition 2.11 *Let $F : J \to J$ be a total mapping. Let $x \in J$ be periodic of order k. Then, the set $\{x, \mathbf{F}(x), \mathbf{F}^2(x), \ldots, \mathbf{F}^{k-1}(x)\}$ is called an* attractor *(of length k) of $\mathbf{F}$.*

A fixed point is also called *singleton attractor*. In the literature, attractors are sometimes also called *limit cycle*. However, in some notions, this involves certain additional stability concepts (cf., e.g., [16]).

Any easy consequence of Proposition 2.8 is that attractors are either disjoint or identical.

Corollary 2.12 *Let* $\mathbf{F} : J \to J$ *be a total map. If* $\{x_1, \ldots, x_\ell\}$ *and* $\{y_1, \ldots, y_r\}$ *are two attractors of* $\mathbf{F}$ *such that* $\{x_1, \ldots, x_\ell\} \cap \{y_1, \ldots, y_r\} \neq \emptyset$ *then* $\{x_1, \ldots, x_\ell\} = \{y_1, \ldots, y_r\}$.

A finite network is a network where each attribute has a finite set of items and a finite range. For finite networks, orbits have a simple structure.

Proposition 2.13 *Let* $F : J \to J$ *be a map on a finite set* J. *Let* $(x_i)_{i\in\mathbb{N}}$ *be the orbit of* $x_0 \in J$ *under* $\mathbf{F}$. *Then, there are* $k_0 \in \mathbb{N}$ *and* $\ell_0 \in \mathbb{N}_+$ *such that*

(i) $\{x_0, \ldots, x_{k_0-1}\}$ *is the set of* k_0 *transient states of the orbit of* x_0 *under* $\mathbf{F}$ *and*

(ii) $\{x_{k_0}, \ldots, x_{k_0+\ell_0-1}\}$ *is an attractor of length* ℓ_0 *of* $\mathbf{F}$.

Proof: Let $(x_i)_{i\in\mathbb{N}}$ be the orbit of $x_0 \in J$ under $\mathbf{F}$, i.e., $x_i = \mathbf{F}^i(x_0)$. Since J is finite, there are $k \geq 0$ and $\ell > 0$ such that $\mathbf{F}^k(x_0) = x_k = x_{k+\ell} = \mathbf{F}^{k+\ell}(x_0)$. Define parameters k_0 and ℓ_0 as follows (in this order):

$$\begin{aligned} k_0 &=_{\text{def}} \min \{ \, k \mid \mathbf{F}^k(x_0) = \mathbf{F}^{k+r}(x_0) \text{ for some } r > 0 \, \} \\ \ell_0 &=_{\text{def}} \min \{ \, r \mid \mathbf{F}^k(x_0) = \mathbf{F}^{k_0+r}(x_0) \, \} \end{aligned}$$

Then, for all $r > 0$, it holds that

$$\mathbf{F}^{k_0+r}(x_0) = \mathbf{F}^r\left(\mathbf{F}^{k_0}(x_0)\right) = \mathbf{F}^r\left(\mathbf{F}^{k_0+\ell_0}(x_0)\right) = \mathbf{F}^{k_0+\ell_0+r}(x_0).$$

Hence, x_i is periodic of order ℓ_0 if $i \geq k_0$, which is tatement (ii), and x_i is transient if $i < k_0$, which is statement (i). This proves the proposition. ■

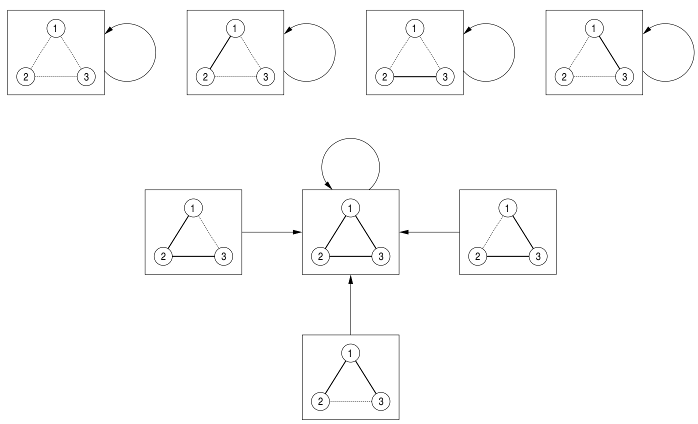
2.2.2 Basins of attraction

In principle, iterated maps can be studied graph-theoretically. A total map $\mathbf{F} : J \to J$ on a finite set J can be associated with the directed graph $\Gamma(\mathbf{F}) = (J, E)$, called *state graph of* $\mathbf{F}$, where $E =_{\text{def}} \{ \, (x, \mathbf{F}(x)) \mid x \in J \, \}$. Note that $\Gamma(\mathbf{F})$ may have loops.

According to Proposition 2.8, Corollary 2.12, and Proposition 2.13, the state graph of $\mathbf{F}$ can be uniquely decomposed into

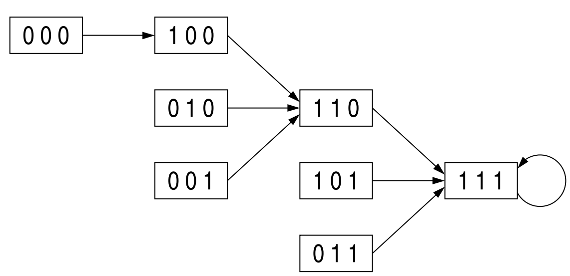
- disjoint cycles $C_1, \ldots, C_k$ (representing attractors) and
- disjoint (directed) trees $T_1, \ldots, T_r$ (representing transient states) each of which is incident with exactly one cycle $C_1, \ldots C_k$

Example: (*Link prediction*) For $n = 3$, the state graph of $\mathbf{F}$ is as follows:



Note that the state graph $\Gamma(\mathbf{F})$ consists of $2^{\binom{n}{2}} = 2^{O(n^2)}$ graphs given the interaction domain $\mathcal{I} = (A \times A) \setminus \{ (i,i) \mid i \in A \}$.

Example: (*Riots*) We only consider a very small population $A = \{1, 2, 3\}$ with given thresholds $t_1 = 0$, $t_2 = 1$, and $t_3 = 2$. Then, the state graph can be visualized as follows (where participation of person i is described the i-th letter):



Note that the state graph contains 2^n vertices.

An attractor together with all its incident trees is called *basin of attraction*.

More formally, let $\mathbf{F} : J \to J$ be a total map on a finite set J. A set $E \subseteq J$ is called *invariant set* if and only if for all $k \in \mathbb{N}$, $\mathbf{F}^k(E) \subseteq E$. Each invariant set contains an attractor. So, J can be uniquely decomposed into r invariant sets where r is the number of attractors of $\mathbf{F}$. A basin of attraction is one component of this decomposition.

It is clear that transient states have visiting probability zero. The following proposition gives the precise visiting probability of a periodic state in terms of the structure of its corresponding basin of attraction.

Proposition 2.14 *Let $\mathbf{F} : J \to J$ be a total map on a finite, non-empty set J. Let $x \in J$ be periodic, and let $E \subseteq J$ be the basin of attraction of (the attractor of) x. Suppose E consists of s transient and r periodic states. Then, the visiting probability of x in a random orbit is*

$$\left(1 + \frac{s}{r}\right) \cdot \frac{1}{\|J\|}.$$

Proof: Let $z \in J$ be an arbitrary state. Consider the orbit $(z_i)_{i\in\mathbb{N}}$ of z under $\mathbf{F}$, i.e., $z = z_0$ and $\mathbf{F}^k(z_0) = z_k$ for all $k > 0$. Suppose $z_0, \ldots, z_{k_0-1}$ are all transient states and $z_{k_0}, \ldots, z_{k_0+r-1}$ are all periodic states (of order r). Suppose that $x \in J$ is a state in the orbit $(z_i)_{i\in\mathbb{N}}$. Define

$$P_x =_{\text{def}} \mathbf{P}\left[x \text{ is visited in } (z_i)_{i\in\mathbb{N}}\right].$$

Then, P_x is given by a frequency sequence of the initial segments of the orbit:

$$P_x = \lim_{N\to\infty} \frac{\|\{\ i \mid i \in \{0, 1, \ldots, N-1\} \text{ and } z_i = x\ \}\|}{N}$$

To calculate P_x, we have two cases. If x is transient, then $P_x = 0$ (since x occurs only once on the orbit). If x is periodic of order r, then the frequency of visiting x in an initial segment of length N approaches $1/r$ for increasing N. Now, consider a periodic $x \in J$ following the specification given in the proposition. Then, x lies on $s + r$ orbits. So, the visiting probability of x is

$$\mathbf{P}[x \text{ is visited in some orbit}] = \frac{s+r}{\|J\|} \cdot \frac{1}{r} = \left(1 + \frac{s}{r}\right) \cdot \frac{1}{\|J\|}.$$

This proves the proposition. ■

Example: We discuss our examples in the light of Proposition 2.14.

- (*Link prediction*) For $n = 3$, it is easily seen from the state graph that all graphs containing no open triangle have equal visiting probability $1/8$ while the complete graph has visiting probability $1/2$. Indeed, we observe that the basin of attraction of the complete graph contains $s = 3$ transient and $r = 1$ periodic graphs.
- (*Riots*) In the scenario with the given thresholds, there is only one fixed point attracting all initial states. Thus, this fixed point (i.e., all persons join the riot) has visiting probability one. Indeed, there $s = 2^n - 1$ transient states and $r = 1$ periodic states.

- (*Job rotation*) In the networks $\{x, \pi\}$ und $\{y, \pi\}$, there are no transient states and all states have equal visiting probability $1/n$ in a corresponding orbit. Note that the job-rotation map is not total as the values of x and y are induced by the initial assignment π. As there are $(n-1)!$ permutations that are n-cycles, the visiting probability of an assignment in some orbit is $1/n!$.

2.3 Stochastic iterated network maps

In the following, we consider iterated random maps $\mathbf{F} : J \to J$. There are two sources of randomness in dynamics: (a) a lack of information on parameters or values of attributes involved in the map, and (b) the use of simulations.

A formal approach to iterated random maps is as follows (cf. [4]). Instead of just one map, an iterated random map is understood as a family of functions, i.e., $\mathbf{F} = \{ f_\omega \mid \omega \in \Omega \}$ where $f_\omega : J \to J$. Here, Ω is a probability space with a probability distribution μ on Ω. For one step of the dynamics, this means that for $x \in J$, we choose an ω according to μ and go to $f_\omega(x)$. In the simplest case, we may assume that μ does not depend on x. We thus obtain a sequence of random variables:

$$X_0 = x_0, \quad X_1 = f_{\omega_1}(x_0), \quad X_2 = f_{\omega_2}(f_{\omega_1}(x_0)), \quad \ldots;$$

inductively: $X_n = f_{\omega_n}(X_{n-1})$ where $\omega_1, \omega_2, \ldots$ are independent choices from Ω.

Example: (*The psychology of conformity*) Suppose we are given an interaction domain $\mathcal{I} = \{ (i,j) \mid i,j \in A, i \neq j \}$ for a set $A = \{1, \ldots, n\}$. Consider a network attribute $x : \mathcal{I} \to \{0,1\}$ which is fixed over time. For $i \in A$ define $N_x(i) = \{ j \in A \mid x_{ij} \neq 0 \}$, i.e., the neighborhood of i in network x. We are interested in the opinion attritbute $o : A \to \{0,1\}$ which changes over time according to the following local rule (cf. [10]): uniformly at random, choose a directed edge (i,j) in the graph of the network x and set o_i to o_j. Globally speaking, the more neighbors of an agent have the same opinion the more likely the agent has the same opinion. This clearly describes an iteration process of a random map in the above sense, i.e., we have a family of maps $\mathbf{F} =_{\mathrm{def}} \{ f_{(i,j)} \mid (i,j) \in E(x) \}$ equipped with the uniform distribution on the set of edges.

2.3.1 Markov chains

The formal approach of stochastic iterated network maps leads naturally to the concept of Markov chains (cf., e.g., [9]).

An infinite sequence $(X_t)_{t\in\mathbb{N}}$ of random variables $X_t : \Omega \to J$, $\|J\| = n$, is called (homogeneous) *Markov chain* with finite state space J iff for all $i, j \in J, z_0, z_1, \ldots, z_{t-1} \in J$,

$$\mathbf{P}\left[X_{t+1} = j | X_t = i, X_{t-1} = z_{t-1}, \ldots, X_1 = z_1, X_0 = z_0\right] = \mathbf{P}[X_{t+1} = j | X_t = i].$$

The matrix $P \in \mathbb{R}^{n\times n}$ defined for all $i, j \in J$ by

$$p_{ij} =_{\mathrm{def}} \mathbf{P}[X_{t+1} = j | X_t = i]$$

is called *transition matrix*. Note that P does not depend on t—the reason why the Markov chain is homogeneous.

Example: (*The psychology of conformity*) In our example, the sequence of opinions $(o^{(t)})_{t\in\mathbb{N}}$ is a homogeneous Markov chain since all the choices of updating edges are independent and do not depend on t. More concretely, let us consider three persons interacting asymmetrically in the following sense: $A = \{1, 2, 3\}$ is the set of agents, $J = \{0, 1\}^3$ is the finite states space, and network x is given by $x_{1,2} = x_{2,1} = 1, x_{2,3} = x_{3,2} = 1$, and $x_{1,3} = x_{3,1} = 0$. That is, $N_x(1) = \{2\}$, $N_x(2) = \{1, 3\}$, and $N_x(3) = \{2\}$. So, agents 1 and 3 do not directly communicate. We have four directed edges, each of which can be chosen with probability 1/4.

We want to calculate transition probablities between states according to our probabilistic update rule. It is clear that $000 \to 000$ and $111 \to 111$ with probability one. For state $001 \in J$, we obtain the following transition probabilities to the possible successor states:

$001 \to 000$	:	$1/4$	edge $(3, 2)$ has to be chosen
$001 \to 001$	:	$1/2$	edges $(1, 2)$ and $(2, 1)$ have to be chosen
$001 \to 010$	:	0	at most one agent can change her opinion
$001 \to 011$	:	$1/4$	edge $(2, 1)$ has to be chosen
$001 \to 100$	:	0	at most one agent can change her opinion
$001 \to 101$	:	0	no neighbor of agent 1 has opinion 1
$001 \to 110$	:	0	at most one agent can change her opinion
$001 \to 111$	:	0	at most one agent can change her opinion

Similarly, all other transition probabilities can be calculated.

The transitions can be represented in 8×8 transition matrix P as follows (when enumerating the state space in lexicographical order):

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/4 & 1/2 & 0 & 1/4 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/4 & 0 & 0 & 1/4 & 0 \\ 0 & 1/4 & 0 & 1/2 & 0 & 0 & 0 & 1/4 \\ 1/4 & 0 & 0 & 0 & 1/2 & 0 & 1/4 & 0 \\ 0 & 1/4 & 0 & 0 & 1/4 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1/4 & 0 & 1/2 & 1/4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Alternatively, Markov chains can be represented sparsely as a weighted graph, where edge weights are just the probabilities:

Let $(X_t)_{t\in\mathbb{N}}$ be a Markov chain over J, $\|J\| = n$. Then, the distribution $q^{(t)} = (q_1^{(t)}, \dots, q_n^{(t)})$ of the Markov chain at time t satisfies for all $i \in J$,

$$q_i^{(t)} = \mathbf{P}[X_t = i];$$

$q^{(0)}$ is the initial distribution. A useful property of Markov chain is that the distribution at time t are obtained as linear iterated map given by the transition matrix P.

Proposition 2.15 *Let $(X_t)_{t\in\mathbb{N}}$ be a Markov chain with state space J, initial distribution $q^{(0)}$, and transition matrix P. For the distribution $q^{(t)}$ at time t, it holds that*

$$q^{(t)} = q^{(0)} P^t.$$

Proof: We prove the statement by induction on t.

- *basis of induction $t = 0$:* Trivial (with $P^0 = I$ where I is the identiy matrix).
- *inductive step $t > 0$:* For $t > 0$ and $j \in J$, we calculate

$$\begin{aligned} q_j^{(t)} &= \mathbf{P}[X_t = j] \\ &= \sum_{i=1}^{n} \mathbf{P}[X_t = j | X_{t-1} = i] \cdot \mathbf{P}[X_{t-1} = i] && \text{(law of total probability)} \\ &= \sum_{i=1}^{n} p_{ij} \cdot \mathbf{P}[X_{t-1} = i] \\ &= \sum_{i=1}^{n} q_i^{(t-1)} \cdot p_{ij} && \text{(by inductive assumption)} \\ &= \left(q^{(t-1)P}\right)_j \end{aligned}$$

This proves the proposition. ∎

2.3.2 The stationary distribution

Definition 2.16 *Let $(X_t)_{t\in\mathbb{N}}$ be a Markov chain with state space J, $\|J\| = n$, and transition matrix P. A distribution π on J is said to be* stationary *if and only if $\pi = \pi \cdot P$.*

Example: (*The pyschology of conformity*) A stationary distribution in our concrete example Markov chain has the form $(q, 0, 0, 0, 0, 0, 0, 1-q)$ for $0 \le q \le 1$. No other distribution are stationary.

Definition 2.17 *Let $(X_t)_{t\in\mathbb{N}}$ be a Markov chain with state space J, $\|J\| = n$, and transition matrix P.*

1. *A state $i \in J$ is said to be* absorbing *if and only if $p_{ij} = 0$ for all $j \neq i$.*
2. *A state $i \in J$ is said to be* transient *if and only if*
$$\mathbf{P}[\text{there exists } t > 0 \text{ such that } X_t = i | X_0 = i] < 1.$$
3. *A state $i \in J$ is said to be* recurrent *if and only if*
$$\mathbf{P}[\text{there exists } t > 0 \text{ such that } X_t = i | X_0 = i] = 1.$$

The classification of the state in the psychology-of-conformity example above can be left to the reader.

Definition 2.18 *Let $(X_t)_{t\in\mathbb{N}}$ be a Markov chain with state space J, $\|J\| = n$, and transition matrix P.*

1. *$(X_t)_{t\in\mathbb{N}}$ is said to be* irreducible *if and only if for all $i, j \in J$, there exists a $t > 0$ such that $(P^t)_{ij} > 0$; otherwise, $(X_t)_{t\in\mathbb{N}}$ is reducible.*
2. *The* period *$d(i)$ for $i \in J$ is defined as*
$$d(i) =_{\text{def}} \gcd \{\ t > 0 \mid (P^t)_{ii} > 0\ \}.$$
If $d(i) = 1$ then i is called aperiodic. *$(X_t)_{t\in\mathbb{N}}$ is said to be* aperiodic *if and only if all states are aperiodic; otherwise, $(X_t)_{t\in N}$ is periodic.*
3. *$(X_t)_{t\in\mathbb{N}}$ is said to be* ergodic *if and only if $(X_t)_{t\in\mathbb{N}}$ is irreducible and aperiodic.*

In other word, a Markov chain is irreducible if and only if the weighted graph of the Markov chain is strongly connected. Each Markov chain with loops is aperiodic.

We state the fundamental theorem for ergodic Markov chains without a proof (the interested reader is pointed to, e.g., [9]):

Theorem 2.19 *For each ergodic Markov chain* $(X_t)_{t\in\mathbb{N}}$*, it holds that*

$$\lim_{t\to\infty} q^{(t)} = \pi$$

independently from the initial distribution $q^{(0)}$*, where* π *is the unique stationary distribution of* $(X_t)_{t\in\mathbb{N}}$*.*

Example: (*The psychology of conformity*) The Markov chain for the small example is clearly not irreducible; however, it can be made ergodic by additing a little random noise, i.e., instead of transition matrix $P \in \mathbb{R}^{n\times n}$, we consider a perturbed matrix P_ε,

$$P_\varepsilon =_{\text{def}} (1-\varepsilon)\cdot P + \frac{\varepsilon}{n}\cdot \mathbf{1}_{n\times n},$$

where $\mathbf{1}_{n\times n}$ denotes the $n\times n$ all-ones matrix. The Markov chain given by P_ε for $0<\varepsilon<$ is always ergodic and, thus, has a unique stationary distribution.

For instance, for the Markov chain above we find as the stationary distribution $\pi = (\pi_1,\ldots,\pi_n)$ for the matrix P_ε:

$$\begin{aligned}
\pi_{000} = \pi_{111} &= \frac{1}{8}\cdot\frac{4+\varepsilon-\varepsilon^2}{1+3\varepsilon}\\
\pi_{001} = \pi_{011} = \pi_{100} = \pi_{110} &= \frac{1}{8}\cdot\frac{5\varepsilon-\varepsilon^2}{1+3\varepsilon}\\
\pi_{010} = \pi_{101} &= \frac{1}{8}\cdot\varepsilon
\end{aligned}$$

Observe that, when ε approaches zero, the limit of P_ε is just P and the limit of the stationary distribution is $(1/2,0,0,0,0,0,0,1/2)$ which is *one* stationary distribution of P.

Network formation 3

We investigate processes by which networks are formed in a stable manner via interaction of rational agents. In the following, we use game theory to analyze such processes.

3.1 Strategic network formation and game theory

As we consider only numerical attributes, we restrict ourselves to the class of games with utility functions.

3.1.1 Games with utilities

Definition 3.1 *A* game with utilities Γ *is a triple* $(A, (S_1, \dots, S_n), (u_1, \dots, u_n))$, *where*

1. $A = \{1, \dots, n\}$ *is a finite, non-empty set of* agents*,*
2. S_i *is a non-empty set of* strategies *of agent* $i \in A$*, and*
3. $u_i : S_1 \times \cdots \times S_n \to \mathbb{R}$ *is a utility function for agent* i*.*

According to the definition above, we introduce some notations:

- $S =_{\text{def}} \bigtimes_{k=1}^{n} S_k$ denotes the set of all strategy profiles of all agents; $S_{-i} =_{\text{def}} \bigtimes_{\substack{k=1 \\ k \neq i}}^{n} S_k$ denotes the set of all strategy profiles of all agents except agent i.
- For a strategy profile $s = (s_1, \dots, s_n) \in S$, let s_{-i} denote the $(n-1)$-tuple consisting of strategies of all agents except agent i, i.e., $s_{-i} = (s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n)$.
- So, $s = (s_i, s_{-i})$ and $S = S_i \times S_{-i}$, by convention.
- We use $u = (u_1, \dots, u_n) : S \to \mathbb{R}^n$ to denote the vector utility function, and we use $u_i(s) = u_i(s_1, \dots, s_n) = u_i(s_i, s_{-i})$ to denote agent's i utility of a strategy profile

We consider a game Γ as a *one-shot non-cooperative game*. Each agent u chooses a strategy $s_i \in S_i$ independently of other agents and without knowing the choices of the other agents. The result is a strategy profile $s = (s_1, \dots, s_n)$. Each agent i evaluates strategy profile s according to the utility function u_i (or, receives payoff $u_i(s)$).

A notion central to game theory is the Nash equilibrium.

Definition 3.2 *Let $\Gamma = (A, S, u)$ be a game with utilities, involving n agents. A strategy profile $s^* = (s_1^*, \ldots, s_n^*)$ is called* Nash equilibrium *if and only if $u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$ for all $s_i \in S_i$ and all $i \in A$.*

Intuitively, in a Nash equilibrium, no agent has an incentive to deviate from the chosen strategy. It captures one possible interpretation of a stable situation.

Example: We exemplify the notions for three standard games (only loosely connected to networks).

- *Battle of sexes*: Male M and Female F want to spend time together, i.e., $A = \{M, F\}$. Alternatives are cinema (c) or football (f). So, the sets of strategies for both are $S_M = S_F = \{c, f\}$. The set of strategy profiles is

$$S = S_F \times S_M = \{\ (c,c), (c,f), (f,c), (f,f)\ \}$$

where the first component of a pair denotes Female's strategy and the second component is Male's strategy. Now, on the one hand-side, Male prefers football over cinema but together is better than alone. So, M's preference can be described by the following utility function:

$$u_M : \begin{array}{rl} (f,f) \mapsto & 3 \\ (c,c) \mapsto & 2 \\ (c,f) \mapsto & 1 \\ (f,c) \mapsto & 0 \end{array}$$

On the other hand-side, Female prefers cinema over football but together is better than alone. So, F's utilities could be as follows:

$$u_F : \begin{array}{rl} (c,c) \mapsto & 3 \\ (f,f) \mapsto & 2 \\ (c,f) \mapsto & 1 \\ (f,c) \mapsto & 0 \end{array}$$

Combined, both utility functions can be modelled as a payoff (bi-)matrix:

$$\begin{array}{cc} & \begin{array}{cc} & M \\ f & c \end{array} \\ F \begin{array}{c} f \\ c \end{array} & \begin{pmatrix} (2,3) & (0,0) \\ (1,1) & (3,2) \end{pmatrix} \end{array}$$

Since all information on the game is contained in this representation, we will also identify such a matrix with a 2-person game.

Which strategy profiles are Nash equilibria? We examine all strategy profiles individually:

- (c, c) is a Nash equilibrium, since

$$u_F(c,c) = 3 > 0 = u_F(f,c)$$
$$u_M(c,c) = 2 > 1 = u_M(c,f)$$

- (c, f) is not a Nash equilibrium, since

$$u_F(c,f) = 1 < 2 = u_F(f,f)$$

- (f, c) is not a Nash equilibrium, since

$$u_M(f,c) = 0 < 3 = u_M(f,f)$$

- (f, f) is a Nash equilibrium, since

$$u_F(f,f) = 2 > 1 = u_F(c,f)$$
$$u_M(f,f) = 3 > 0 = u_M(f,c)$$

Now, suppose Female is more decisive: she excludes football an option. Thus, F's modified utility function leads to the following (bimatrix) game

$$\begin{pmatrix} (1,3) & (0,0) \\ (2,1) & (3,2) \end{pmatrix}$$

Then, the only Nash equilibrium is (c, c).

- *Prisoner's dilemma*: Bonnie and Clyde have been captivated and charged with bank robbery. However, the prosecutor is only able to prove illegal possession of firearms to them; without confessions, the sentence will then be 3 years in prison. If one of them makes a confession then the confessor will be sentenced to one year and the non-confessor will be sentenced to 9 years in prison. If both confess then they will be sentenced to 7 years in prison, respectively.

  A game-based formulation of this decision scenario is given by the following game with utilities:

$$\begin{array}{cc} & \begin{array}{cc} s_{21} & s_{22} \end{array} \\ \begin{array}{c} s_{11} \\ s_{12} \end{array} & \begin{pmatrix} (-7,-7) & (-1,-9) \\ (-9,-1) & (-3,-3) \end{pmatrix} \end{array}$$

  where s_{i1} stands for strategy "confession" and s_{i2} stands for "no confession."

  Which strategy profiles are Nash equilibria?

  - (s_{11}, s_{21}) is a Nash equilibrium, since

$$u_1(s_{11}, s_{21}) = -7 > -9 = u_1(s_{12}, s_{21})$$
$$u_2(s_{11}, s_{21}) = -7 > -9 = u_2(s_{11}, s_{22})$$

- (s_{11}, s_{22}) is not a Nash equilibrium, since

$$u_2(s_{11}, s_{22}) = -9 < -7 = u_2(s_{11}, s_{21})$$

- (s_{12}, s_{21}) is not a Nash equilibrium, since

$$u_1(s_{12}, s_{21}) = -9 < -7 = u_1(s_{11}, s_{21})$$

- (s_{12}, s_{22}) is not a Nash equilibrium, since

$$u_1(s_{12}, s_{22}) = -3 < -1 = u_1(s_{11}, s_{22})$$

Why is this game a dilemma? Because (s_{12}, s_{21}) would be a better strategy profile for both. But it is no equilibrium; each agent could be better off when changing the strategy. The reason for that is the lack of communication and coordination.

- *Rock-paper-scissor*: The scenario consists of two players each of them chooses one of the three gestures "rock", "paper", or "scissor" as a strategy. The rules of winning the game are as follows:
  - rock defeats scissor
  - scissor defeats paper
  - paper defeats rock

  The loser of a game pays a unit to the winner. We can express this a game with utilities by the following bimatrix game:

$$\begin{array}{r c} & \begin{array}{ccc} \text{rock} & \text{paper} & \text{scissor} \end{array} \\ \begin{array}{r} \text{rock} \\ \text{paper} \\ \text{scissor} \end{array} & \begin{pmatrix} (0,0) & (-1,1) & (1,-1) \\ (1,-1) & (0,0) & (-1,1) \\ (-1,1) & (1,-1) & (0,0) \end{pmatrix} \end{array}$$

  Obviously, there is no Nash equilibrium for this game in pure strategies.

Example: Consider a group $A = \{1, \dots, n\}$ of n persons being pairwise friends. A person $i \in A$ wants to spend time with each friend $j \in A$ solely but has only limited amount t_i of spare time available. The problem is how to distribute the time among the friends.

We formulate the problem as a game $\Gamma = (A, S, u)$:

- $A = \{1, \dots, n\}$
- $S = S_1 \times \cdots \times S_n$ where

$$S_i =_{\text{def}} \{ \ (s_{i1}, \dots, s_{in}) \mid s_{ij} \geq 0 \text{ and } s_{i1} + \cdots + s_{in} = t_i \ \}$$

- $u = (u_1, \dots, u_n)$ where

$$u_i(s_1, \dots, s_n) =_{\text{def}} \sum_{\substack{j=1 \\ i \neq j}}^{n} \min\{s_{ij}, s_{ji}\}$$

The interpretation of the utility function u is that the mutual time of two persons depends on time reciprocally made available and time spended without a friend is worthless. Persons aim at maximizing time with friends.

Which strategies are Nash equilibria? An easy analysis shows that each symmetrical strategy profile is a Nash equilibrium. A strategy profile s is symmetrical iff $s_{ij} = s_{ji}$ for all $i, j \in A$. Indeed, let $s = (s_1, \dots, s_n) \in S$ be a symmetrical strategy profile, let $i \in A$, and let $s'_i \in S_i$. Then, we obtain,

$$u_i(s'_i, s_{-i}) = \sum_{\substack{j=1 \\ i \neq j}}^{n} \min\{s'_{ij}, s_{ji}\} \leq \sum_{\substack{j=1 \\ i \neq j}}^{n} s_{ji} = \sum_{\substack{j=1 \\ i \neq j}}^{n} \min\{s_{ij}, s_{ji}\} = u_i(s_i, s_{-i}).$$

Among the symmetrical strategy profiles is the strategy profile where all persons spend their time alone, which is of course highly stable as no person has time for another person. Thus, in general, stability is not the only possible perspective on network formation. Another perspective is efficiency. For instance, we could define "social welfare" of the friendship network as

$$U(s) =_{\mathrm{def}} \sum_{i=1}^{n} u_i(s),$$

i.e., the total mutual time.

3.1.2 The best-response map

An alternative characterization of Nash equilibria can be given by best-response dynamics.

Definition 3.3 *Let $\Gamma = (A, S, u)$ be a game with utilities.*

1. *The* best response *(map) $\beta_i : S_{-i} \to \mathcal{P}(S_i)$* for agent *$i \in A$ is defined by*

$$\beta_i(s_{-i}) =_{\mathrm{def}} \left\{ s_i \in S_i \;\middle|\; u_i(s_i, s_{-i}) = \max_{s'_i \in S_i} u_i(s'_i, s_{-i}) \right\}$$

2. *The* best response *$\beta : S \to \mathcal{P}(S_1 \times \cdots \times S_n)$ is defined by*

$$\beta(s) =_{\mathrm{def}} \beta_1(s_{-1}) \times \cdots \times \beta_n(s_{-n}).$$

Theorem 3.4 *Let $\Gamma = (A, S, u)$ be a game with utilities. For all $s^* \in S$, it holds*

$$s^* \text{ is a Nash equilibrium} \iff s^* \in \beta(s^*).$$

Proof: Let $s^* \in S$ be a strategy profile. Then, the following chain of equivalences holds:

$$\begin{aligned}
& s^* \text{ is a Nash equilibrium} \\
\iff\; & u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*) \text{ for all } i \in A,\ s_i \in S_i && \text{(by Definition 3.2)} \\
\iff\; & u_i(s_i^*, s_{-i}^*) = \max_{s_i \in S_i} u_i(s_i, s_{-i}^*) \text{ for all } i \in A,\ s_i \in S_i \\
\iff\; & s_i^* \in \beta_i(s_{-i}^*) \text{ for all } i \in A && \text{(by Definition 3.3.1)} \\
\iff\; & s^* \in \beta(s^*) && \text{(by Definition 3.3.2)}
\end{aligned}$$

This proves the theorem. ■

Example: Consider the following payoff matrix for a two-person game with identical utility function

$$\begin{array}{cc}
 & \begin{array}{cc} s_{21} & s_{22} \end{array} \\
\begin{array}{c} s_{11} \\ s_{12} \end{array} & \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}
\end{array}$$

The best responses for the agents are

$$\begin{aligned}
\beta_1(s_{21}) &= \{s_{11}, s_{12}\}, & \beta_1(s_{22}) &= \{s_{11}\} \\
\beta_2(s_{11}) &= \{s_{22}\}, & \beta_2(s_{12}) &= \{s_{22}\}
\end{aligned}$$

So, the best response is:

$$\begin{aligned}
\beta(s_{11}, s_{21}) &= \{s_{11}, s_{12}\} \times \{s_{22}\} \\
\beta(s_{11}, s_{22}) &= \{s_{11}\} \times \{s_{22}\} \\
\beta(s_{12}, s_{21}) &= \{s_{11}, s_{12}\} \times \{s_{22}\} \\
\beta(s_{12}, s_{22}) &= \{s_{11}\} \times \{s_{22}\}
\end{aligned}$$

By Theorem 3.4, (s_{11}, s_{22}) is a unique Nash equilibrium.

3.1.3 The connections model

The connections model is a game-theoretic model for the formation of a static network [11, 19]. We only consider the basic version of the model.

First, we define utility functions on given networks. Let $A = \{1, \ldots, n\}$ be a set of agents, let $\mathcal{I} = A \times A \setminus \{\ (i,i) \mid i \in A\ \}$ be the interaction domain, and let $x : \mathcal{I} \to \{0,1\}$ be a network. Let $G = G(x)$ be the undirected graph of network x. Each agent $i \in A$ receives payoff $0 < \delta < 1$ for each direct link in G. An agent $i \in A$ additionally receives a payoff for indirected links; this (exponentially decreasing) payoff is $\delta^{d(i,j)}$ where $d(i,j)$ is the length of a shortest path in G. Note that $\delta^{d(i,j)} = 0$ for $d(i,j) = \infty$. Each agent $i \in A$ pays a cost $c > 0$ for maintaining a directed link. We call (A, δ, c) a *connection model*. The utility function $u_i(G)$ for graph G is thus given by:

$$u_i(G) = \sum_{i \neq j} \delta^{d(i,j)} \quad - \sum_{\{i,j\} \in E(G)} c$$

Strategically, an agent could keep or drop an existing directed link to an agent. This affects, however, both agents joint by the link. Thus, we need a notion of stability appropriate for this situation.

Definition 3.5 *Let (A, δ, c) be a connection model. An undirected graph $G = (A, E)$ is said to be* stable *if and only if the following conditions are fulfilled:*

1. *For all $\{i, j\} \in E$: $u_i(G) \geq u_i(G - \{i, j\})$.*
2. *For all $\{i, j\} \notin E$: $u_i(G + \{i, j\} - iG - jG) > u_i(G) \Rightarrow u_j(G + \{i, j\} - iG - jG) < u_j(G)$.*

Here, kG denotes an edge set $kG \subseteq \{\ e \in E \mid k \in e\ \}$ for $k \in A$.

Proposition 3.6 *Let (A, δ, c) be a connection model. Then, a stable graph exists. Further,*

1. *if $c \geq \delta$ then the empty graph is stable,*
2. *if $c < \delta$ and $(\delta - c) \leq \delta^2$ then a $K_{1,n-1}$ is stable,*
3. *if $c < \delta$ and $(\delta - c) > \delta^2$ then the K^n is stable.*

Proof: We prove all cases individually.

1. Follows easily from $u_i(A, \{i, j\}) = \delta - c \leq 0 = u_i(A, \emptyset)$.
2. Let $G = (A, E) = K_{1,n-1}$. Let $\{i, j\} \in E$, i.e., i is the center of the star, j is a leaf. Then, we have the following:

$$\begin{aligned}
u_i(K_{1,n-1}) &= \delta \cdot (n-1) - c \cdot (n-1) = (\delta - c) \cdot (n-1) \\
u_i(K_{1,n-1} - \{i, j\}) &= \delta \cdot (n-2) - c \cdot (n-2) = (\delta - c) \cdot (n-2) \\
u_j(K_{1,n-1}) &= \delta - c + \delta^2 \cdot (n-2) \\
u_j(K_{1,n-1} - \{i, j\}) &= 0
\end{aligned}$$

Since $\delta - c > 0$, we obtain the inequalities $u_i(K_{1,n-1}) \geq u_i(K_{1,n-1} - \{i, j\})$ and $u_j(K_{1,n-1}) \geq u_j(K_{1,n-1} - \{i, j\})$, i.e., the first condition of stability is satisfied. Now, let $\{i, j\} \notin E$, i.e., i and j are leaves in graph G. Notice that this implies $n \geq 3$. Then, we obtain:

$$\begin{aligned}
u_i(K_{1,n-1} + \{i, j\} - iG - jG) &\leq u_i(K_{1,n-1} + \{i, j\}) && \text{(since } \delta > c\text{)} \\
&= 2\delta + \delta^2 \cdot (n-3) - 2c \\
&\leq \delta - c + \delta^2 \cdot (n-2) && \text{(since } \delta - c \leq \delta^2\text{)} \\
&= u_i(K_{1,n-1})
\end{aligned}$$

Hence, the second condition of stability is satisfied. Therefore, $K_{1,n-1}$ is stable.

3. Follows easily from $u_i(K^n) = (\delta - c) \cdot (n-1) > (\delta - c) \cdot (n-2) + \delta^2 = u_i(K^n - \{i,j\})$.

This proves the proposition. ■

We want to study whether and which stable networks are reached in a dynamic network formation model Let (A, δ, c) be some connection model. Initially, the graph $G = G_0$ is empty. We consider discrete time steps $T = \mathbb{N}_+$ and a sequence $(G_t)_{t \in T}$ of graphs. The agents are assumed to be *myopic*, i.e., they make decisions as better responses, if possible. More specifically, choose a dyad $\{i,j\}$ uniformly at random at each time step $t \in T$:

- if $\{i,j\} \in E(G_{t-1})$ then either i or j can sever the link
- if $\{i,j\} \notin E(G_{t-1})$ then i and j can form a link $\{i,j\}$ and simultaneously sever any of their other links, if both agents agree.

Proposition 3.7 *Let (A, δ, c) be a connection model. In the network formation process,*

1. *if $(\delta - c) > \delta^2 > 0$ then every link forms and remains,*
2. *if $\delta - c < 0$ then no link ever forms.*

Proof: We give individual arguments for both statements.

1. Assume $\delta - c > \delta^2$. That is, $\delta - c > \delta^2 > \delta^3 > \cdots > \delta^{n-1}$. Thus, each agent prefers a direct link over any indirect link. Suppose i and j are chosen in step t: if $\{i,j\} \notin E(G_{t-1})$ then each agent gains at least $\delta - c - \delta^{d_{G_{t-1}}(i,j)} > 0$ from forming a link; if $\{i,j\} \in E(G_{t-1})$ then, severing the link, an agent's utlity decreases. Thus, a direct link is never broken.
2. Since the graph is initially empty, forming the first link gives payoff $\delta - c < 0$, so will never be formed.

This proves the proposition. ■

Corollary 3.8 *Let (A, δ, c) be a connection model. Then, the following holds:*

1. *If $\delta - c > \delta^2$ then the network formation process converges to K^n.*
2. *If $\delta - c < 0$ then the network formation process converges to the empty graph.*

The remaining case is covered by the following proposition, the proof of which can be found in [19].

Proposition 3.9 *Let* (A, δ, c) *be a connection model,* $\|A\| \geq 4$*. Suppose* $0 < \delta - c < \delta^2$*. Then,*

$$0 < \mathbf{P}[\textit{sequence } (G_t)_{t \in T} \textit{ converges to a } K_{1,n-1}] = O(\tfrac{1}{n}).$$

3.2 Strategic network formation with structural holes

3.2.1 Bridges and structural holes

3.2.2 The model

In the forthcoming, we study a formulation of strategic network formation with structural holes based on game theory [12].

Let $A = \{1, \ldots, n\}$ be a set of agents. Let $S = S_1 \times \cdots \times S_n$ be the set of strategy profiles where

$$S_i = \mathcal{P}(\{\ (i,j) \mid j \in A \setminus \{i\}\ \}),$$

for each $i \in A$, i.e., i's strategy is basically a set of selected persons; here, friendship is considered as a directed relationship which needs not necessarily be mutually confirmed. Let $c_{i,j} \geq 0$ denote agent i's cost of buying a link to agent j. The utility function $u = (u_1, \ldots, u_n)$ is given as follows for each $i \in A$:

$$u_i(s_1, \ldots, s_n) =_{\mathrm{def}} \alpha_0 \cdot (\|s_i\| + \{\ j \mid (j,i) \in s_j\ \}\|) + \sum_{\substack{(i,j),(i,k) \in s_i \\ j \neq k}} \beta(r_{j,k}) - \sum_{(i,j) \in s_i} c_{i,j},$$

where $\alpha_0 \geq 0$ is the benefit of a direct link, β is a decreasing, non-negative function representing the intermediary benefit of each agent in the middle of a length-2 path, and $r_{j,k}$ is the number of length-2 paths (we set $r_{j,k} = 0$ if there is a direct link between j and k in either direction) in the underlying undirected graph induced by the strategies of the agents.

3.2.3 Existence of equilibria

We want to analyze how possible equilibrium networks look like and whether equilibria exist at all. We first identify a (sub-)class of equilibrium graphs of specific directed, multipartite structure: let $q = \lfloor n/k \rfloor$ for number n of vertices, number k of parties, i.e., there are q independent sets of size k, and one independent set of size $l \equiv n \bmod k$. Then, $G_{n,k}$ is a complete, multipartite graph with vertex set

$$V = V_1 \cup V_2 \cup \cdots \cup V_q \cup V_{q+1}$$

where $V_i \cap V_j \neq \emptyset$ for $i \neq j$, $\|V_1\| = \cdots = \|V_q\| = k$, $\|V_{q+1}\| = l$, and edge set

$$E =_{\text{def}} \{\ (u,v) \mid u \in V_i, v \in V_j,\ \text{and } j < i\ \},$$

i.e., all edges are directed from higher-number parties to lower-number parties.

Lemma 3.10 *Let $G = (V, E)$ be an undirected graph containing an independent set $I \subseteq V$ of size k such that all vertices $v \notin I$ are adjacent to all vertices i.*

1. *The change in utility for vertex $v \notin I$ from deleting all edges to I is*
$$B(n,k) =_{\text{def}} k \cdot (\alpha_0 - 1) + \binom{k}{2}\beta(n-k).$$
2. *If $B(n,k) \geq 0$ then vertex $v \notin I$ will keep all edges to I.*
3. *If $B(n,k) < 0$ then vertex $v \notin I$ will drop all edges to I.*

Proof: To prove the first statement, we calculate the total value of all edges to I. The benefit of direct links is $k \cdot (\alpha_0 - 1)$, the intermediary benefit for a single pair of vertices in i is $\beta(n-k)$. Thus, the overall value is

$$k \cdot (\alpha_0 - 1) + \binom{k}{2} \cdot \beta(n-k) {}_{\text{def}} = B(n,k)$$

For the second and the third statements, suppose $v \notin I$ has edges to some set $A \subseteq I$ of size k'. Then, the value of these edges is

$$k' \cdot (\alpha_0 - 1) + \binom{k'}{2} \cdot \beta(n-k)$$

which is maximized for $k' = 0$ or $k' = k$ (as it is a convex function in k'). The statements follow. This proves the lemma. ■

Theorem 3.11 *For each set of n agents there exists a $k \in \mathbb{N}_+$ such that the graph $G_{n,k}$ is a Nash equilibrium.*

Proof: We make a case distinction. If $B(n,1) = \alpha_0 - 1 \geq 0$ then $\alpha_0 \geq 1$ and the complete graph $G_{n,1}$ is a Nash equilibrium (by Lemma 3.10.2). If $B(n, n-1) < 0$ then the empty graph $G_{n,n}$ is a Nash equilibrium (by Lemma 3.10.3). If $B(n,1) < 0$ and $B(n, n-1) \geq 0$ there exists a number $k$such that $B(n,k) \geq 0$ and $B(n, k-1) < 0$. Since $G_{n,k}$ is a complete, multipartite, agent $i \in V_j$ cannot gain intermediary benefit from V_ℓ, $\ell \neq i$, by only connecting from inside V_j. Since $B(n,k) \geq 0$, agent i keeps all edges to V_ℓ for $\ell < j$. Since $B(n, k-1) < 0$, agent i will have no edges to $V_j \setminus \{i\}$. Thus, $G_{n,k}$ is a Nash equilibrium. This proves the theorem. ■

Theorem 3.12 *For every choice of $\alpha_0 > 0$ and function β such that $\beta(r) \geq \gamma \cdot r^{-1}$ for some constant $\gamma > 0$, every equilibrium graph has at least $\Omega(n^2)$ edges.*

3.3 Network congestion and potential games

In the following, we should have a traffic scenario in mind. Suppose we are given a circuit graph C_4 with vertex set $V = \{A, B, C, D\}$ and edge set consisting of the edges $\{A, B\}, \{B, C\}, \{C, D\}$, and $\{D, A\}$. Furthermore, suppose an agent located in A wishes to select a shortest route to D and an agent sitting in B wishes to select a shortest route to C. If both agents use the same link then the congestion (or latency) increases. Of course, the agents aim at minimizing their cost. In the scenario, it is certainly interesting to know which traffic network x is stable and whether there always stable networks. Note that the network attribute x (on the interaction domain C_4) is formed by the number of agents use the same link. Such scenarios with strategic agents can be studied using the notion of *potential games* as introduced by Monderer and Shapley [15]. This is an important class of games with equilibrium guarantee.

3.3.1 Potential functions

Definition 3.13 *Let $\Gamma = (A, S, u)$ be a game with utilities, and let $P : S \to \mathbb{R}$ be any function.*

1. *P is said to be an* ordinal potential function *for Γ if and only if for all $i \in A$, $s_{-i} \in S_{-i}$, $s_i, \bar{s}_i \in S_i$,*

$$u_i(s_i, s_{-i}) - u_i(\bar{s}_i, s_{-i}) > 0 \iff P(s_i, s_{-i}) - P(\bar{s}_i, s_{-i}) > 0.$$

   Γ is said to be an ordinal potential game *if and only if there is an ordinal potential function for Γ.*

2. *P is said to be a* potential function *for Γ if and only if for all $i \in A$, $s_{-i} \in S_{-i}$, $s_i, \bar{s}_i \in S_i$,*

$$u_i(s_i, s_{-i}) - u_i(\bar{s}_i, s_{-i}) = P(s_i, s_{-i}) - P(\bar{s}_i, s_{-i}).$$

   Γ is said to be a potential game *if and only if there is a potential function for Γ.*

Example: We discuss the notions for two games.

- Consider the following bimatrix game:

$$\Gamma = \begin{pmatrix} (0,3) & (1,2) \\ (3,1) & (2,0) \end{pmatrix}$$

  According to Definition 3.13, it suffices to consider the following differences:

$$\begin{aligned} u_1(s_{11}, s_{21}) - u_1(s_{12}, s_{21}) &= -3 \\ u_1(s_{11}, s_{22}) - u_1(s_{12}, s_{22}) &= -1 \\ u_2(s_{11}, s_{21}) - u_2(s_{11}, s_{22}) &= 1 \\ u_2(s_{12}, s_{21}) - u_2(s_{12}, s_{22}) &= 1 \end{aligned}$$

Then, Γ is an ordinal potential game. An ordinal potential function P is represented by the matrix

$$\Gamma = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix},$$

since

$$\begin{array}{rcl} P(s_{11}, s_{21}) - P(s_{12}, s_{21}) & = & -1 \; < \; 0 \\ P(s_{11}, s_{22}) - P(s_{12}, s_{22}) & = & -1 \; < \; 0 \\ P(s_{11}, s_{21}) - P(s_{11}, s_{22}) & = & 1 \; > \; 0 \\ P(s_{12}, s_{21}) - P(s_{12}, s_{22}) & = & 1 \; > \; 0 \end{array}$$

However, Γ is not a potential game. (An explanation will be given later.)

- Recall that the *prisoner's dilemma* can be represent by the following bimatrix game:

$$\Gamma = \begin{pmatrix} (-7,-7) & (-1,-9) \\ (-9,-1) & (-3,-3) \end{pmatrix}$$

Γ is a potential game, where the potential function P is given by

$$P = \begin{pmatrix} 4 & 2 \\ 2 & 0 \end{pmatrix}.$$

Proposition 3.14 *Let $\Gamma = (A, S, u)$ be a game with utilities and an ordinal potential function P, and let $s^* \in S$. Then, s^* is a Nash equilibrium if and only if for all $i \in A$ and $s_i \in S_i$, it holds that*

$$P(s^*) \geq P(s_i, s^*_{-i}).$$

Proof: Immediate from Definition 3.13. ■

Corollary 3.15 *Each finite ordinal potential game has a Nash equilibrium.*

Proof: For a finite ordinal potential game Γ, it's ordinal potential funciton P has a maximum. Let $s^* \in S$ be such that $P(s^*)$ is maximum. Then, s^* is a Nash equilibrium by Proposition 3.14. ■

Though we have a certain flexibility in choosing an ordinal potential for an ordinal potential game, in case of exact potentials it is less so: they are unique up to some additive constant.

Proposition 3.16 *Let $\Gamma = (A, S, u)$ be a potential game with potentials P_1 and P_2. Then, there is a $c \in \mathbb{R}$ such that for all $s \in S$,*

$$P_1(s) - P_2(s) = c$$

Proof: Choose any $s^* \in S$. Define for all $s \in S$,

$$H(s) =_{\text{def}} \sum_{i=1}^{m} \left(u_i(t^{i-1} - u_i(t^i)\right),$$

where $t^0 = s$ and $t^i = (s_i^*, t_{-i}^{i-1})$ for $i \in \{1, \ldots, m\}$. For each potential P for Γ we have

$$H(s) = \sum_{i=1}^{m} \left(u_i(t^{i-1}) - u_i(t^i)\right) = \sum_{i=1}^{m} \left(P(t^{i-1}) - P(t^i)\right) = P(t^0) - P(t^m) = P(s) - P(s^*).$$

Hence,

$$P_1(s) - P_2(s) = H(s) + P_1(s^*) - (H(s) + P_2(s^*)) = P_1(s^*) - P_2(s^*).$$

The last difference is constant. This proves the proposition. ■

3.3.2 Characterizations of potential games

How can we decide whether a given game with utilites is, in fact, a potential game? To answer this question, we give a characterization based on the structure of utility functions. It is helpful to introduce some additional notions.

Let $\Gamma = (A, S, u)$ be a game with utilites.

A sequence $p = (s^0, s^1, \ldots, s^N)$ is a *path* in Γ if and only if for all $k \geq 1$, there is an $i \in A$ such that $s^k = (s_i, s_{-i}^{k-1})$ for some $s_i \in S_i$ with $s_i \neq s_i^{k-1}$. The agent $i \in A$ is then called the *deviator* for k. A path $p = (s^0, s^1, \ldots, s^N)$ is said to be *closed* iff $s^0 = s^N$. A path $p = (s^0, s^1, \ldots, s^N)$ is said to be *simple* iff $s^j \neq s^k$ for all $0 \leq j < k \leq N - 1$.

Furthermore, for a finite path $p = (s^0, s^{,} \ldots, s^N)$ in Γ, define

$$I(\Gamma, p) =_{\text{def}} \sum_{k=1}^{N} \left(u_{i_k}(s^k) - u_{i_k}(s^{k-1})\right),$$

where i_k is the deviator for k.

Theorem 3.17 *Let $\Gamma = (A, S, u)$ be a game with utilities. The following statements are equivalent:*

1. *Γ is a potential game.*
2. *$I(\Gamma, p) = 0$ for each finite, closed path p in Γ.*
3. *$I(\Gamma, p) = 0$ for each finite, simple, closed path p in Γ.*
4. *$I(\Gamma, p) = 0$ for each finite, simple, closed path p in Γ of length 4.*

Proof: We show the following implications:

- (1) ⇒ (2): Let P be a potential function for $\Gamma = (A, S, u)$. Let $p = (s^0, s^1, \ldots, s^N)$ be a closed path. Then, we conclude

$$\begin{aligned} I(\Gamma, p) &= \sum_{k=1}^{N} \Big(u_{i_k}(s^k) - u_{i_k}(s^{k-1}) \Big) \\ &= \sum_{k=1}^{N} \Big(P(s^k) - P(s^{k-1}) \Big) \qquad \text{(since } P \text{ is a potential function for } \Gamma) \\ &= P(s^N) - P(s^0) \\ &= 0 \qquad \text{(since } s^N = s^0) \end{aligned}$$

- (2) ⇒ (1): Fix an arbitrary strategy profile $z \in S$. For $s \in S$, let $p(s) = (s^0, \ldots, s^N)$ denote an arbitrary path from $s^0 = z$ to $s^N = s$. We define

$$P(s) =_{\text{def}} I(\Gamma, p(s)).$$

Note that there is always a path from z to a strategy profile s. We have to show that the following two statements are true:

1. P is well-defined, i.e., the definition of P is independent of the choice of the path $p(s)$.
2. P is a potential function for Γ

This can be seen as follows:

1. Let $q(s) = (s^0, \ldots t^M)$ be another path such that $t^0 = z$ and $t^M = s$. Then, the concatenated path $\gamma = (s^0, \ldots, s^N, t^{M-1}, \ldots, t^0)$ is a closed path in Γ. By our assumption, it holds that $I(\Gamma, \gamma) = 0$. We conclude

$$\begin{aligned} I(\Gamma, p(s)) &= -I\left(\Gamma, (s^N, t^{M-1}, \ldots, t^0)\right) \\ &= I\left(\Gamma, (t^0, \ldots, t^{M-1}, s^N)\right) \\ &= I(\Gamma, q(s)) \end{aligned}$$

2. For $i \in A$, let $s_i, s_i' \in S_i$ be two strategies, let $s_{-i} \in S_{-i}$. Again by our assumption, we obtain

$$\begin{aligned} 0 &= I\left(\Gamma, \left((s_i, s_{-i}), \ldots, z, \ldots, (s_i', s_{-i}), (s_i, s_{-i})\right)\right) \\ &= I\left(\Gamma, ((s_i, s_{-i}), \ldots, z)\right) + I\left(\Gamma, \left(z, \ldots, (s_i', s_{-i}), (s_i, s_{-i})\right)\right) \\ &= -I\left(\Gamma, (z, \ldots, (s_i, s_{-i}))\right) + I\left(\Gamma, \left(z, \ldots, (s_i', s_{-i})\right)\right) + \\ &\qquad + u_i(s_i, s_{-i}) - u_i(s_i', s_{-i}) \end{aligned}$$

Consequently,

$$\begin{aligned} u_i(s_i, s_{-i}) - u_i(s_i', s_{-i}) &= I(\Gamma, (z, \ldots, (s_i, s_{-i}))) - I(\Gamma, (z, \ldots, (s_i', s_{-i}))) \\ &= P(s_i, s_{-i}) - P(s_i', s_{-i}) \end{aligned}$$

Hence, P is a potential function.

- (2) $\Rightarrow$ (3): Trivial.
- (3) $\Rightarrow$ (4): Trivial.
- (4) $\Rightarrow$ (2): Suppose $I(\Gamma, p) = 0$ for all simple, closed paths p of length 4 in $\Gamma = (A, S, u)$. We show that $I(\Gamma, p) = 0$ for all closed paths p of length N in Γ by induction on N:
  - *base of induction* $N \leq 4$: Cases $N \in \{1, 2, 3\}$ are trivial (in particular, there are no closed paths of odd lengths); for $N = 4$, the statement holds by the assumption.
  - *inductive step* $N > 4$: Let $p = (s^0, s^1, \ldots, s^N)$ be a closed path with $N \geq 5$. Let $(i_1, \ldots, i_N)$ be the sequence of deviators for each step, i.e., $s^j = (s_{i_j}, s^{j-1}_{-i_j})$ such that $s_{i_j} \neq s^{j-1}_{i_j}$. Without loss of generality, assume $i_1 = 1$. Since $s^N = s^0$, there is $2 \leq j \leq N$ such that $i_j = 1$ and $s^j_{i_j} = s^0_1$. Choose j to be minimal subject to this condition, i.e., there is no $2 \leq k < j$ satisfying $i_k = 1$ and $s^k_{i_k} = s^0_1$.
    First, suppose $j = 2$. That is, $s^2 = s^0$. Consider the path $q =_{\text{def}} (s^2, \ldots, s^N)$ of length $N - 1$. Then,

$$\begin{aligned} I(\Gamma, p) &= I(\Gamma, q) + u_1(s^2) - u_1(s^1) + u_1(s^1) - u_1(s^0) \\ &= u_1(s^2) - u_1(s^0) && \text{(by inductive assumption)} \\ &= 0 && \text{(since } s^2 = s^0\text{)} \end{aligned}$$

    Now, suppose $j \geq 3$, i.e, $j \in \{3, \ldots, N\}$. Then, we have two subcases:
    1. Subcase $i_{j-1} = i_j$. Consider path $q =_{\text{def}} (s^0, \ldots, s^{j-2}, s^j, \ldots, s^N)$. It holds

$$\begin{aligned} & I(\Gamma, q) \\ = \; & I\left(\Gamma, (s^0, \ldots, s^{j-2})\right) + u_{i_j}(s^j) - u_{i_j}(s^{j-2}) + I\left(\Gamma, (s^j, \ldots, s^N)\right) \\ = \; & I\left(\Gamma, (s^0, \ldots, s^{j-2})\right) + u_{i_j}(s^j) - u_{i_j}(s^{j-1}) + \\ & \qquad + u_{i_j}(s^{j-1}) - u_{i_j}(s^{j-2}) + I\left(\Gamma, (s^j, \ldots, s^N)\right) \\ = \; & I\left(\Gamma, (s^0, \ldots, s^{j-2})\right) + u_{i_j}(s^j) - u_{i_j}(s^{j-1}) + \\ & \qquad + u_{i_{j-1}}(s^{j-1}) - u_{i_{j-1}}(s^{j-2}) + I\left(\Gamma, (s^j, \ldots, s^N)\right) \\ & \qquad \text{(since } i_{j-1} = i_j\text{)} \\ = \; & I\left(\Gamma, (s^0, \ldots, s^{j-2}, s^{j-1}, s^j, \ldots, s^N)\right) \\ = \; & I(\Gamma, p) \end{aligned}$$

       Thus, by the inductive assumption, $I(\Gamma, p) = I(\Gamma, q) = 0$.
    2. Subcase $i_{j-1} \neq i_j$. That is, we have the following scenario: ... Define a path $q_j = \text{def}(s^0, \ldots, s^{j-2}, t^{j-1}, s^j, \ldots, s^N)$ where $t^{j-1} = (s_{i_j}, s^{j-2}_{-i_j})$, i.e., the deviator in $(j-1)$-st step is 1. Now, path $r =_{\text{def}} (s^{j-2}, t^{j-1}, s^j, s^{j-1}, s^{j-2})$

is simple (since $i_j \neq i_{j-1}$), closed, and has length 4. Hence, $I(\Gamma, r) = 0$. That is,

$$I\left(\Gamma, (s^{j-2}, s^{j-1}, s^j)\right) = I\left(\Gamma, (s^{j-2}, t^{j-1}, s^j)\right).$$

Therefore, $I(\Gamma, p) = I(\Gamma, q_j)$.

Recursively repeated, we obtain a sequence of paths $q_j, q_{j-1}, \ldots, q_3$ such that $I(\Gamma, p) = I(\Gamma, q_k)$ for all $k \in \{3, \ldots, j\}$ and the deviator in q_k's step $k-1$ is 1. The path q_3 corresponds to the case $j = 2$ above. Thus,

$$I(\Gamma, p) = I(\Gamma, q_3) = 0.$$

This proves the theorem. ■

We want to characterize potential games from a dynamical perspective.

Let $\Gamma = (A, S, u)$ be a game with utilities. Let $(s^t)_{t \in I}$ be any finite or infinite sequence of strategy profiles, i.e., $I = \mathbb{N}$ or $I = \{0, 1, \ldots, n\}$ for some $n \in \mathbb{N}$. Then, the sequence $(s^t)_{t \in I}$ is called an *improvement path* if and only if for all $t \in I, t > 0$, there is an $i \in A$ such that $s^t \neq s^{t-1}$, $(s^t)_{-i} = (s^{t-1})_{-i}$, and $u_i(s^t) > u_i(s^{t-1})$. The intuition behind this definition is that each deviator choose a better alternative. Γ is said to have the *Finite Improvement Property* (FIP) if and only if every improvement path is finite.

To establish our characterization, some technical limitations on games are required: A game $\Gamma = (A, S, u)$ is called *degenerate* iff there exist $i \in A$, $s_i, s'_i \in S_i$, $s_i \neq s'_i$, and $s_{-i} \in S_{-i}$ such that $u_i(s_i, s_{-i}) = u_i(s'_i, s_{-i})$; otherwise, Γ is called *nondegenerate*.

Theorem 3.18 *Let Γ be a finite, nondegenerate game with utilities. Then, Γ has the FIP if and only Γ is an ordinal potential game.*

Proof:

($\Leftarrow$): Let $\Gamma = (A, S, u)$ be a finite game with ordinal potential function P, i.e., for all $i \in A$, $s_i, s'_i \in S_i$, $s_{-i} \in S_{-i}$,

$$u_i(s'_i, s_{-i}) \geq u_i(s_i, s_{-i}) \Longleftrightarrow P(s_i, s_{-i}) \geq P(s'_i, s_{-i}).$$

Let $\gamma = (s^0, s^1, s^2, \ldots)$ be an improvement path, and let $(i_1, i_2, \ldots)$ be the sequence of γ's deviators. Then, for all $t \in I, t > 0$, it holds that $u_{i_t}(s^t) > u_{i_t}(s^{t-1})$. Hence, $P(s^0) < P(s^1) < P(s^2) < \ldots$. As S is a finite set, $\gamma = (s^0, s^1, s^2, \ldots)$ is a finite sequence, i.e., $\|I\| < \infty$.

($\Rightarrow$): Let $\Gamma = (A, S, u)$ have the FIP. Define a binary relation $>$ on S:

$$s > s' \Longleftrightarrow_{\text{def}} s \neq s' \text{ and there is an improvement path from } s \text{ to } s'$$

Since Γ has the FIP, $>$ is a strict order relation on S, i.e., $>$ is irreflexive and transitive. Any finite strict order can be represented by a function: A set $Z \subseteq S$

is *represented* iff there is a mapping $Q : Z \to \mathbb{R}$ such that for all $s, s' \in Z$, $s > s'$ implies $Q(s) > Q(S')$. Let Z^* be a maximal, represented subset of S.

We show $Z^* = S$. To the contrary, assume there is an $x \in S$, $x \notin Z^*$. Then, there are three (possibly overlapping) cases:

1. There is no $z \in Z*$ such that $z > x$. Define an extension $Q' : Z^* \cup \{x\} \to \mathbb{R}$ by:

$$Q'(z) = \begin{cases} Q(z) & \text{if } z \in Z^* \\ \max\{\ Q(z) \mid z \in Z^*\ \} + 1 & \text{if } z = x \end{cases}$$

   Q' represents $Z^* \cup \{x\}$ and, thus, contradicts to the maximality of Z^*.

2. There is no $z \in Z*$ such that $z < x$. Dually to the first case, define an extension $Q' : Z^* \cup \{x\} \to \mathbb{R}$ by:

$$Q'(z) = \begin{cases} Q(z) & \text{if } z \in Z^* \\ \min\{\ Q(z) \mid z \in Z^*\ \} - 1 & \text{if } z = x \end{cases}$$

   Q' represents $Z^* \cup \{x\}$ and, thus, contradicts to the maximality of Z^*.

3. For some $z, z' \in Z*$, it holds that $z > x > z'$. In this case, define an extension $Q' : Z^* \cup \{x\} \to \mathbb{R}$ by:

$$Q'(z) = \begin{cases} Q(z) & \text{if } z \in Z^* \\ \frac{1}{2}(\max\{\ Q(z) \mid z < x\ \} + \min\{\ Q(z) \mid z > x\ \}) & \text{if } z = x \end{cases}$$

   Q' represents $Z^* \cup \{x\}$ and, thus, contradicts to the maximality of Z^*.

Therefore, $Z^* = S$.

Let Q represent S. Then, Q is an ordinal potential function: Suppose $s_i, s_i' \in S_i$, $s_{-i} \in S_{-i}$. Then, $u_i(s_i, s_{-i}) \neq u_i(s_i', s_{-i})$ since Γ is nondegenerate. So, without loss of generality, $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$. Thus, $(s_i, s_{-i}) > (s_i', s_{-i})$. (Note there is an improvement path of length one.) Hence, $Q(s_i, s_{-i}) > Q(s_i', s_{-i})$.

This proves the theorem. ∎

3.3.3 Congestion games

We come back to the traffic scenario that motivates the study of potential games. A game-theoretic formulation of such scenarios is known as *congestion games* (introduced in economics in [18].)

We can analyze this scenario as a game with utilities

$$\Gamma =_{\text{def}} (\{\mathtt{A}, \mathtt{B}\}, \{\{1,2\}, \{3,4\}\} \times \{\{1,3\}, \{2,4\}\}, u)$$

where the utility function $u = (u_1, u_2)$ is given by the following bimatrix:

$$\begin{array}{cc} & \begin{array}{cc} \{1,3\} & \{2,4\} \end{array} \\ \begin{array}{c} \{1,2\} \\ \{3,4\} \end{array} & \begin{pmatrix} (c_1(2)+c_2(1), c_1(2)+c_3(1)) & (c_1(1)+c_2(2), c_2(2)+c_4(1)) \\ (c_3(2)+c_4(1), c_1(1)+c_3(2)) & (c_3(1)+c_4(2), c_2(1)+c_4(2)) \end{pmatrix} \end{array}$$

There are two simple, closed paths of length 4 in the game Γ. So, let p be the one in counter-clockwise direction starting with the upper left strategy profile corner, and let q be the one in clockwise direction also starting with the upper left strategy profile corner. It holds that

$$\begin{aligned} I(\Gamma, p) &= \underbrace{c_3(2)+c_4(1)-c_1(2)-c_2(1)}_{\texttt{A}\text{ deviates}} + \overbrace{c_2(1)+c_4(2)-c_1(1)-c_3(2)}^{\texttt{B}\text{ deviates}} + \\ &\quad + \underbrace{c_1(1)+c_2(2)-c_3(1)-c_4(2)}_{\texttt{A}\text{ deviates}} + \overbrace{c_1(2)+c_3(1)-c_2(2)-c_4(1)}^{\texttt{B}\text{ deviates}} \\ &= 0 \end{aligned}$$

Since $I(\Gamma, q) = -I(\Gamma, p) = 0$, we obtain from Theorem 3.17 that Γ is a potential game.

Definition 3.19 *A* congestion model *is a tuple* $(A, F, (S_i)_{i \in A}, (w_f)_{f \in F})$ *such that*

1. $A = \{1, \ldots, n\}$ *is a non-empty, finite set of* agents *(routers),*
2. F *is a non-empty, finite set of* facilities *(links),*
3. $S_i \subseteq \mathcal{P}(F)$ *is a non-empty set of strategies (routes) for each agent* $i \in A$, *and*
4. $w_f : \{1, \ldots, n\} \to \mathbb{R}$ *is a* cost *(wealth, latency)* function *for each facility* $f \in F$*; if* k *agents choose* f *then the cost for each agent is* $w_f(k)$.

Definition 3.20 *Let* $(A, F, (S_i)_{i \in A}, (w_f)_{f \in F})$ *be a congestion model. Then,* $\Gamma = (A, (S_i)_{i \in A}, u)$ *is called* congestion game *if and only if for all* $i \in A$, $s = (s_i, s_{-i}) \in S$,

$$u_i(s) = \sum_{f \in s_i} w_f(\sigma_f(s)),$$

where $\sigma_f(s) = \|\{i \in A | f \in s_i\}\|$.

Without proof we state the following theorem which shows that potential games and congestion games are essentially the same class of finite games.

Theorem 3.21 1. *Each congestion game is a potential game.*

2. *Each potential game is isomorphic to a congestion game.*

The proof of the first statement relies on the ROSENTHAL *potential*:

$$P(s) =_{\text{def}} \sum_{f \in \bigcup_{i \in A} s_i} \sum_{k=1}^{\sigma_f(s)} w_f(k)$$

Opinion Formation 4

4.1 Attitudes and attitude change

An *attitude* is a positive, negative, or mixed reaction to a person, object, or idea. Attitudes are (often) measured using (multi-item) questionnaires known as *attitude scales*, e.g., Likert scales. Note that attitude scales are based on the principle of forced choices. An *opinion* is the result of selecting a value from an attribute range D, i.e., $o : A \to R$ is a behavioral attribute. In other words, an opinion $o_i \in R$ of actor $i \in A$ is the state of i in the population A. Note that opinions are specific for an experimental design.

In the forthcoming, we consider attitude change by persuasive communication. Among the many models developped in social psychology, a standard model is the Elaboration Likelihood Model (ELM), a dual-process model by Petty and Cacioppo [17]. It makes an antagonistic distinction between two ways to persuasion depending on the so-called *need for cognition* of an audience receiving a message from a source:

- *central route* to persuasion: actor thinks carefully about a communication and is influenced by the strength of its arguments.
- *peripheral route* to persuasion: actor does not think carefully about a communication and is influenced by superficial cues.

Typically, persuasive communication varies between these antagonistic poles, thus forming a spectre of routes. Models of opinion dynamics implement mechanisms of interdependent influence by persuasive communication affecting opinion changes, and by this presumably attitude changes, of the actors.

4.2 Reaching a Consensus

The basic mechanism in opinion formation is consensus dynamics. For instance, consider a group of n individuals who must act together as a team or committee. Each individual in the group has an opinion o_i (or, more generally, a subjective probability distribution depending on some paramter ϑ). The question is: when and how does the group reach a consensus, i.e., an opinion profile $o = (o_1, \dots, o_n)$ satisfying $o_1 = \cdots = o_n$?

A baseline model that, on the one hand-side, aims at explanation for consensus and, on the other hand-side, can be used as a protocol for reaching a consensus was proposed by DeGroot [3].

4.2.1 Consensus in opinion pools

An opinion pool is basically an iterated map $P : R^A \to R^A$, i.e., given an opinion profile o, $P(o)$ is the updated opinion profile. Classic opinion pools are the *linear opinion pools* [5, 3], which describe the opinion profile for all actors after k rounds of opinion updates as a weighted average:

$$o_i^{(k+1)} =_{\text{def}} \sum_{j=1}^{n} p_{ij} o_j^{(k)}$$

for $1 \leq i \leq n$, $k \geq 0$, $p_{ij} \geq 0$, and $\sum_{j=1}^{n} p_{ij} = 1$. Further types of opinion pools have been studied since then, e.g., g-quasi-linear opinion pools or logarithmic opinion pools (cf., e.g., [7, 13]). We only consider linear opinion pools.

A linear opinion pool can be represented by a matrix $P \in \mathbb{R}^{n\times n}$ where entries are exactly the weights p_{ij} of the opinion pool. Thus, the iteration of P induces orbits on initial opinions $o^{(0)}$ which can be describe by matrix multiplication:

$$o^{(k)} = P \cdot o^{(k-1)} = P^2 \cdot o^{(k-2)} = \cdots = P^k \cdot o^{(0)}$$

Definition 4.1 *Let $A = \{1, \ldots, n\}$ be a set of actors, $P \in \mathbb{R}^{n\times n}$ be a linear opinion pool, and $o^{(0)}$ be some initial opinion profile. An opinion $o^* \in \mathbb{R}$ is said to be a* consensus for P and $o^{(0)}$ *if and only if for all $i \in A$,*

$$\lim_{k\to\infty} o_i^{(k)} = o^*.$$

That is, in order to obtain a consensus, all individual opinion orbits must converge to the same opinion. It easily follows that a consensus profile $(o^*, \ldots, o^*)$ is a fixed point for P. This leads to the question of which conditions must be satisfied by P to guarantee the existence of a consensus no matter which initial opinion profile is given.

Proposition 4.2 *Let $A = \{1, \ldots, n\}$ be a set of actors and let $P \in \mathbb{R}^{n\times n}$ be a linear opinion pool. For all initial opinion profiles $o^{(0)}$, there exists a consensus o^* (depending on the initial opinion profile) if and only if there exists a vector $\pi = (\pi_1, \ldots, \pi_n)$ such that $\lim_{k\to\infty}(P^k)_{ij} = \pi_j$ for all $i \in A$. In case that the latter condition is true, the consensus is given by $o^* = \sum_{i=1}^{n} \pi_i o_i^{(0)}$.*

Note that the criterion given in the proposition is not true if we fix some initial opinion profile. Though the criterion remains sufficient for the existence of a consensus, it is no longer necessary. For instance, the all-zero vector $(0, \ldots, 0)$ is a consensus profile for each matrix P. For a given initial opinion profile there are characterizations of the existence of consensus profiles depending on *both* the linear opinion pool P and the initial opinion profile (see [1]). We now prove Proposition 4.2 (with credits to Julian Vill).

Proof: The direction ($\Leftarrow$) is easily seen by the specification of the consensus profile. For ($\Rightarrow$), assume that all orbits converge into a consensus profile, i.e., for all initial opinion profiles $o^{(0)}$, we have $\lim_{k\to\infty} o_i^{(k)} = \lim_{k\to\infty} \left(P^k \cdot o^{(0)}\right)_i = o^*$ for some o^* and all $i \in A$. Hence, for each opinion profile x, it holds that

$$\lim_{k\to\infty} \left(P^k \cdot x\right)_r = \lim_{k\to\infty} \left(P^k \cdot x\right)_s \tag{4.1}$$

for all $r, s \in A$. Let e_j denote the opinion profile where all opinions are set to zero except those for actor j which is set to one. So, $P^k \cdot e_j$ is just the j-th column of the matrix P^k. Since $\lim_{k\to\infty} \left(P^k \cdot e_j\right) = \left(\lim_{k\to\infty} P^k\right) \cdot e_j$, we obtain from Eq. (4.1) that all entries in the j-th column of the limit matrix are equal. Define $\pi_j =_{\mathrm{def}} \lim_{k\to\infty} \left(P^k \cdot e_j\right)_1$. It follows that for all $i, j \in A$,

$$\pi_j = \lim_{k\to\infty} \left(P^k \cdot e_j\right)_1 = \lim_{k\to\infty} \left(P^k \cdot e_j\right)_i = \lim_{k\to\infty} (P^k)_{ij}$$

This proves the proposition. ■

4.2.2 Conditions of convergence

In the light of Proposition 4.2 we are interested in further criteria for convergence to consensus. Since P satisfies $\sum_{j=1}^{n} p_{ij} = 1$, P is a stochastic matrix describing a Markov chain. Then, p_{ij} is simply the probability that agent i adopts the opinion of agent j.

The following theorem which is stated without a proof is representative for results originating in the analysis of stationary distributions and mixing behavior of Markov chains.

Theorem 4.3 *Let $P \in \mathbb{R}^{n\times n}$ be a linear opinion pool. If there exists a $k \in \mathbb{N}_+$ such that every element in at least one column of P^k is positive then a consensus will be reached.*

Example: Suppose we have a group of $n = 2$ agents, and

$$P = \begin{pmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{pmatrix}$$

According to the theorem above, a consensus will be reached. Indeed, the stationary distribution is calculated via

$$\begin{aligned} \pi_1 &= \frac{1}{2} \cdot \pi_1 + \frac{1}{4} \cdot \pi_2 \\ \pi_2 &= \frac{1}{2} \cdot \pi_1 + \frac{3}{4} \cdot \pi_2 \end{aligned}$$

So, $\pi = (1/3, 2/3)^T$ and the consensus is $1/3 \cdot o_1 + 2/3 \cdot o_2$.

Now, suppose a third individual joins the group, so that

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/4 & 3/4 & 0 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$$

Here, the stationary distribution is $(1/3, 2/3, 0)$, so the third individual has no weight in the consensus.

4.3 The Friedkin-Johnsen model

So far, it is assumed that no outside data, observations, or information is available, i.e., we studied completely endogenous dynamics. Now, consider the following generalized linear model described by an opinion orbit (as proposed by Friedkin and Johnsen [6]):

$$\begin{aligned} o^{(0)} &= X_0 B_0 \text{ where} \\ &\qquad X_0 \in \mathbb{R}^{n\times m} \text{ represents weighted influence of } m \text{ exogenous variables} \\ &\qquad B_0 \in \mathbb{R}^{m\times 1} \text{ represents values of } m \text{ exogenous variables} \end{aligned}$$

and for $k > 0$,

$$\begin{aligned} o^{(k)} &= \alpha_k W_k o^{(k-1)} + \beta_k X_k B_k \text{ where} \\ &\qquad X_k \in \mathbb{R}^{n\times m}, B_k \in \mathbb{R}^{m\times 1} \text{ have the same meaning as above} \\ &\qquad W_k \in \mathbb{R}^{n\times n} \text{ represents the network of influence} \\ &\qquad\qquad \text{(where } 0 \le w_{ij}^{(k)} \le 1 \text{ and } \textstyle\sum_{j=1}^{n} w_{ij}^{(k)} = 1) \\ &\qquad \alpha_k \in \mathbb{R} \text{ represents a weight on the endogenous conditions} \\ &\qquad \beta_k \in \mathbb{R} \text{ represents a weight on the exogenous conditions} \end{aligned}$$

So, in the general model, for each time step we have different influences of the variables and in the networks. This makes the model almost unanalyzable. If we impose the homogeneity assumptions

$$\begin{aligned} X_0 = X_1 = X_2 = \cdots &= X \\ B_0 = B_1 = B_2 = \cdots &= B \\ W_1 = W_2 = \cdots &= W \\ \alpha_1 = \alpha_2 = \cdots &= \alpha \\ \beta_1 = \beta_2 = \cdots &= \beta \end{aligned}$$

on the variables, we come up with a basic model defining an iterated linear map

$$F \;:\; D^n \to D^n \;:\; x \;\mapsto\; \alpha W x + \beta X B$$

on some opinion domain D.

Let us consider the orbit of $x^{(0)} = XB$ under F:

$$\begin{aligned}
x^{(0)} &= XB \\
x^{(1)} &= F(x^{(0)}) = \alpha WXB + \beta XB = (\alpha W + \beta I)XB \\
x^{(2)} &= F(x^{(1)}) = \alpha W(\alpha W + \beta I)XB + \beta XB = (\alpha^2 W^2 + \alpha\beta W + \beta I)XB \\
x^{(3)} &= F(x^{(2)}) = \alpha W(\alpha^2 W^2 + \alpha\beta W + \beta I)XB + \beta XB \\
&= (\alpha^3 W^3 + \alpha^2\beta W^2 + \alpha\beta W + \beta I)XB
\end{aligned}$$

An easy inductive argument shows that for the k-th iteration F^k:

$$x^{(k)} = \alpha^k W^k XB + \left(\sum_{t=0}^{k-1} \alpha^t W^t\right) \beta XB$$

Lemma 4.4 *Let $0 < \alpha < 1$. For all $k \in \mathbb{N}_+$ the following statements hold:*

1. $\lim_{k\to\infty} \alpha^k W^k = 0$
2. $\lim_{k\to\infty} \sum_{t=0}^{k} \alpha^t W^t = (I - \alpha W)^{-1}$

Proof: We prove both equation individually.

1. Recall that all entries of the matrix W (and thus of matrices W^k, too) are nonnegative and at most one. So, for $i, j \in \{1, \ldots, n\}$ we have

$$0 \le \lim_{k\to\infty} \left(\alpha^k W^k\right)_{ij} = \lim_{k\to\infty} \alpha^k w_{ij}^{(k)} \le \lim_{k\to\infty} \alpha^k = 0.$$

2. Since W is a stochastic matrix, the maximum eigenvalue is one. It follows that the inverse matrices $(I - \alpha W)^{-1}$ always exists. We calculate

$$\begin{aligned}
\left(\lim_{k\to\infty} \sum_{t=0}^{k} \alpha^t W^t\right) \cdot (I - \alpha W) &= \lim_{k\to\infty} \left(\sum_{t=0}^{k} \alpha^t W^t\right) \cdot (I - \alpha W) \\
&= \lim_{k\to\infty} \left(\sum_{t=0}^{k} \alpha^t W^t - \sum_{t=0}^{k} \alpha^{t+1} W^{t+1}\right) \\
&= \alpha^0 W^0 - \lim_{k\to\infty} \alpha^{k+1} W^{k+1} \\
&= I \qquad \text{(by statement 1.)}
\end{aligned}$$

This proves the lemma. ■

By this lemma, we have a strong convergence guarantee for all orbits in the basic Friedkin-Johnsen model.

Theorem 4.5 *If* $0 < \alpha < 1$ *then*

$$\lim_{k\to\infty} x^{(k)} = (I - \alpha W)^{-1} \beta X B,$$

where $XB = x^{(0)}$ *is the initial opinion profile.*

Bibliography

[1] R. L. Berger. A necessary and sufficient condition for reaching a consensus using DeGroot's method. *Journal of the Americal Statistical Association*, 76(374):415–418, 1981.

[2] G. Deffuant, F. Amblard, G. Weisbuch, and T. Faure. How can extremism prevail? A study on the relative agreement interaction model. *Journal of Artificial Societies and Social Simulation*, 5(4), 2002.

[3] M. H. DeGroot. Reaching a consensus. *Journal of the Americal Statistical Association*, 69(345):118–121, 1974.

[4] P. Diaconis and D. Freedman. Iterated random functions. *SIAM Review*, 41(1):45–76, 1999.

[5] J. R. P. French. A formal theory of social power. *Psychological Review*, 63(3):181–194, 1956.

[6] N. E. Friedkin and E. C. Johnsen. Social influence and opinions. *Journal of Mathematical Sociology*, 15:193–205, 1990.

[7] G. L. Gilardoni and M. K. Clayton. On reaching a consensus using DeGroot's iterative pooling. *Annals of Statistics*, 21(1):391–401, 1993.

[8] M. S. Granovetter. Threshold models of collective behavior. *American Journal of Sociology*, 83(6):1420–1443, 1978.

[9] O. Häggström. *Finite Markov Chains and Algorithmic Applications*, volume 52 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2002.

[10] H.-B. Hu and X.-F. Wang. Discrete opinion dynamics on networks based on social influence. *Journal of Physics A: Mathematical and Theoretical*, 42(22):225005, 2009.

[11] M. O. Jackson and A. Wolinsky. A strategic model of social and economic networks. *Journal of Economic Theory*, 71(1):44–74, 1996.

[12] J. M. Kleinberg, S. Suri, É. Tardos, and T. Wexler. Strategic network formation with structural holes. In *Proceedings of the 9th ACM Conference on Electronic Commerce (EC'08)*, pages 284–293. ACM Press, New York, NY, 2008.

[13] K. Lehrer and C. G. Wagner. *Rational Consensus in Science and Society*. Reidel, Dordrecht, 1981.

[14] D. Liben-Nowell and J. M. Kleinberg. The link-prediction problem for social networks. *Journal of the Association for Information Science and Technology*, 58(7):1019–1031, 2007.

[15] D. Monderer and L. S. Shapley. Potential games. *Games and Economic Behavior*, 14:124–143, 1996.

[16] H. S. Mortveit and C. M. Reidys. *An Introduction to Sequential Dynamical Systems*. Springer, New York, 2008.

[17] R. E. Petty and J. T. Cacioppo. The elaboration likelihood model of persuasion. In L. Berkowitz, editor, *Advances in experimental social psychology*, volume 19, pages 123–205. Academic Press, New York, NY, 1986.

[18] R. W. Rosenthal. A class of games possessing pure-strategy Nash equilibria. *International Journal of Game Theory*, 2(1):65–67, 1973.

[19] A. Watts. A dynamic model of network formation. *Games and Economic Behavior*, 34(2):331–341, 2001.